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ON SOME RELATIONS BETWEEN THE EULER CLASS GROUP OF REAL VARIETIES AND TOPOLOGY

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ABSTRACT. The Euler class group of a smooth affine n-dimensional variety is in a certain sense the analogue of the "nth cohomology group" of the variety. In this paper, we study the Euler class groups of smooth real affine varieties and the connections that these groups have with topology.

1. Introduction. Let A be a Noetherian ring and $J \subset A$ be an ideal such that J/J^2 is generated by n elements. Then, it is known (see for example, [**1**, Lemma 3.2]) that J can be generated by $n+1$ elements. In general, however, J need not be generated by n elements.

For example, if m is the maximal ideal of the coordinate ring of the real circle, corresponding to a real point, then m/m^2 is generated by one element but m is not generated by one element, for the graph of any function which intersects the circle transversally at one point must cross the circle elsewhere.

One therefore poses the following general problem. Let A be a Noetherian ring. Let $J \subset A$ be an ideal such that J/J^2 is generated by n elements. When is J generated by n elements?

In view of the example of the coordinate ring of the circle, the special case of the general problem where the height of the ideal J is equal to the dimension of A is of interest and one poses (again) the following:

Question 1. Let A be a Noetherian ring of dimension n. Let $J \subset A$ be an ideal of height n. Suppose J/J^2 is generated by n elements. What is the obstruction to J being generated by n elements?

We briefly outline the answer to this question:

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Let J be an ideal of height n in a Noetherian ring A of dimension $n > 1$. Assume that J/J^2 is generated by n elements. An orientation ω_J of J is roughly a set of n generators of J/J^2 modulo a certain equivalence. To the pair (J, ω_J), (where ω_J is an orientation of J), we assign in [**3**] an element of a certain group $E(A)$ (called the Euler class group of A). If this element of $E(A)$ is zero, then J can be generated by n elements, thus giving the desired obstruction.

The study of the Euler class groups $E(A)$ of A was initiated by Nori who defined these groups in the case where A is smooth. If A is a smooth affine domain of dimension $n > 1$ and P is an "oriented" projective module of rank n and trivial determinant, Nori associated to P an element $E(A)$ called the Euler class of P and conjectured that P splits a summand of rank 1 if the Euler class of P vanishes. (This conjecture which provided the impetus to study the group $E(A)$, was proved in [**1**].)

The example of the coordinate ring of the real circle leads us to believe that, for real affine varieties of dimension n, the obstruction to J being generated by n elements in Question 1 is partly topological. Since this obstruction lies in $E(A)$, it is of interest to compute the Euler class groups of real affine varieties. In [**2**], we studied the Euler class groups of smooth real affine varieties of dimension > 1 and computed their structure in terms of topological and algebraic data.

In this paper, we continue this study and explore some connections between the Euler class groups of smooth real affine varieties and differential topology.

We briefly outline the contents of this paper.

In Section 2, we state various versions of Swan's Bertini theorem. This theorem serves as an algebraic substitute for Sard's theorem in differential topology. In Section 3 we motivate the extension of the definitions of Euler class groups of smooth varieties of dimension > 1 to the Euler class groups of curves. In Section 7, we define homomorphisms from the Euler class groups of curves to free abelian groups of a certain rank extending certain results of [**2**] to curves. Sections 2 to 6 contain the necessary preliminaries needed for this extension. In Section 8, we show how the Euler class of a rank 2 projective module over the coordinate ring of the two sphere can be defined by a topological method. We apply this method to prove the known nontriviality of the Hopf bundle and tangent bundle of the real two sphere.

In order to make the paper self-contained and easy to read we have included the proofs of a number of standard results. We also do not state results in their maximum generality, preferring to prove some illuminating special cases.

2. Some preliminaries. In this section we state Swan's Bertini theorem ([**6**, Theorem 1.3 and 1.4]). We deduce various corollaries which will be used in later sections. We begin with a weaker version of Swan's Bertini theorem.

Proposition 2.1. *Let A be a Noetherian ring and $[a_1, a_2, \ldots, a_n, a] \in A^{n+1}$. Then, there exists an element $[b_1, \ldots, b_n] \in A^n$ such that if $I = (a_1 + ab_1, \ldots, a_n + ab_n)$, then we have height $(I_a) \geq n$, that is, if $Q \in \operatorname{Spec} A$, $I \subset Q$ and $a \notin Q$, then* height $(Q) \geq n$.

Proof. If a belongs to every minimal prime ideal of A, then a belongs to every prime ideal of A and there is nothing to prove. Let us suppose that this is not the case, and let $Q_1, \ldots, Q_r$ be the minimal prime ideals of A which do not contain a. Since $a \notin Q_i$, the ideal (a_1, a) is not contained in Q_i, $1 \leq i \leq s$. Hence, we can choose $b_1 \in A$ so that $a_1 + b_1 a \notin \cup_{r=1}^{s} Q_i$.

Having chosen $b_1, \ldots, b_s$ for $s < n$ we choose b_{s+1} as follows:

Suppose $Q'_1, \ldots, Q'_t$ are the minimal prime ideals containing the ideal $(a_1 + b_1 a, \ldots, a_s + b_s a)$ such that $a \notin Q'_i$, $1 \leq i \leq t$. If no such prime ideal exists with the above property, we choose $b_{s+1} = 0$ and $b_i = 0$, $i \geq s+1$. Otherwise, we choose $b_{s+1} \in A$ such that $a_{s+1} + b_{s+1} a \notin \cup_{i=1}^{t} Q'_i$. It is easy to see that the elements $b_1, \ldots, b_n$ so chosen satisfy the required property. $\square$

The above proposition motivates the following Bertini theorem due to Swan. (See [**2**, Theorem 2.11] for the following version.)

Theorem 2.2. *Let A be a geometrically reduced affine ring over an infinite field, and let P be a projective A-module of rank n. Let $(\alpha, a) \in P^* \oplus A$. Then, there exists an element $\beta \in P^*$ such that if $I = (\alpha + a\beta)(P)$ then*

(i) *Either $I_a = A_a$ or I_a is an ideal of height n such that $(A/I)_a$ is a geometrically reduced ring.*

(ii) *If A is smooth $(A/I)_a$ is also smooth.*

(iii) *In particular, if $J = (a_1, a_2, \ldots, a_n, a)$ is an ideal of A, there exist $b_1, \ldots, b_n \in A$ such that if $I = (a_1 + b_1 a, \ldots, a_n + b_n a)$, then I satisfies properties* (i) *and* (ii). $\square$

We now record various corollaries of this theorem for future use in this paper. We state the results only in the generality that we need them.

Corollary 2.3. *Let A be a regular affine domain over an infinite perfect field of dimension n. Let $a_1, \ldots, a_n \in A$ be such that the ideal $(a_1, \ldots, a_n) = \cap_{i=1}^s m_i$, where the m_i, $1 \le i \le s$, are distinct maximal ideals of A. Then, we can choose $b_1, b_2, \ldots, b_{n-1} \in A$ such that the ideal $I = (a_1 + b_1 a_n, a_2 + b_1 a_n, \ldots, a_{n-1} + b_{n-1} a_n)$ satisfies the property that A/I is regular of dimension* 1.

Proof. By Swan's Bertini theorem, we can choose $b_1, \ldots, b_{n-1} \in A$ such that the ideal $I = (a_1 + b_1 a_n, \ldots, a_{n-1} + b_{n-1} a_n)$ satisfies the property that the localization $(A/I)_m$ is regular for every maximal ideal of A which contains I and does not contain a_n. Since $(a_1, \ldots, a_n) = \cap_{i=1}^s m_i$, it follows that the localization $(A/I)_{m_i}$ is regular for every m_i, $1 \le i \le s$. Hence A/I is regular. $\square$

The proof of the following corollary is similar and hence is omitted.

Corollary 2.4. *Let A be a regular affine domain over an infinite perfect field of dimension n. Let $a_1, \ldots, a_n \in A$ be such that the ideal of $(a_1, \ldots, a_n) = J_1 \cap m_2 \cap \cdots \cap m_k$, where $m_2, \ldots, m_k$ are distinct maximal ideals of A and J_1 is m_1-primary, where m_1 is a maximal ideal of A different from $m_2, \ldots, m_k$. Then, we can choose $b_1, \ldots, b_{n-1} \in A$*

such that the ideal $I = (a_1+b_1a, \dots, a_{n-1}+b_{n-1}a)$ *satisfies the property that* $\dim A/I = 1$ *and* $(A/I)_m$ *is regular for every maximal ideal* m *of* A/J *different from* m_1.

Corollary 2.5. *Let* A *be a regular affine domain over an infinite perfect field of dimension* n. *Let* P *be a projective* A-*module of rank* n. *Then, there exists a surjection* $\delta : P \twoheadrightarrow J$, *where either* $J = A$, *or* $J = \cap_{i=1}^{s} m_i$ *is the intersection of finitely many maximal ideals of* A.

Proof. We choose any linear map $\alpha : P \to A$ and apply Swan's Bertini theorem to the element $(\alpha, 1) \in P^* \oplus A$. □

The following corollary has a similar proof.

Corollary 2.6. *Let* A *be a regular affine domain over an infinite perfect field of dimension* n. *Let* P *be a projective* A-*module of rank* n. *Let* $(\alpha, a) \in P^* \oplus A$ *be such that* $\alpha(P) + aA = A$. *Then, there exists a* $\beta \in P^*$ *such that* $(\alpha+a\beta)(P) = \cap_{i=1}^{s} m_i$, *where* m_i *are distinct maximal ideals of* A.

Corollary 2.7. *Let* A *be a regular affine domain over an infinite perfect field of dimension* n. *Let* $J \subset A$ *be an ideal such that* J/J^2 *is generated by* n *elements. Let* $\overline{a_1}, \dots, \overline{a_n}$ *generate* J/J^2. *Then, we can choose lifts* $c_1, \dots, c_n \in J$ *of* $\overline{a_1}, \dots, \overline{a_n}$, *which satisfy the property that* $(c_1, \dots, c_n) = J \cap J'$, *where* $J + J' = A$ *and either* $J' = A$ *or* J *is the intersection of finitely many maximal ideals of* A.

Proof. Since $(a_1, \dots, a_n) + J^2 = J$, we can choose $a \in J^2$ such that $(a_1, \dots, a_n, a) = J$. By Swan's Bertini theorem, we can choose $b_1, \dots, b_n \in A$ such that the ideal $I = (a_1 + b_1a, \dots, a_n + b_na)$ satisfies property (i) of Theorem 2.2. Let $c_i = a_i + b_ia$, $1 \le i \le n$. Then, since $a \in J^2$, $(c_1, \dots, c_n) + J^2 = J$. Hence, $I = (c_1, \dots, c_n) = J \cap J'$, where $J + J' = A$. Note that, since $a \in J$, $(A/I)_a = (A/J')_a$. Hence, it follows that J' satisfies the required property.

3. The Euler class group of a circle. The graph of a polynomial function $f(X, Y) = 0$ of two variables can be thought of as the path of a particle. Let us consider such a function and assume that the particle crosses the X axis transversally at every point where the graph meets the X axis.

Let us assume that the X axis is a boundary which separates "land" from "water." Let us choose in a continuous manner a tangent vector at each point on the X axis and also at each point on the graph of f. At each point where the graph of the function meets the X axis, we then have a basis of $\mathbf{R}^2$. The determinant of the matrix formed by these two basis vectors is a nonzero real number. The sign of the determinant is either positive or negative depending on whether the particle is "entering water from land" or "entering land from water."

In the above situation, we considered the zeroes of two functions $f(X, Y) = 0$ and $Y = 0$ (the X axis). Let us generalize this a little bit and consider the graphs of the zeroes of two arbitrary polynomial functions $f(X, Y) = 0$ and $g(X, Y) = 0$. Assume that when the graphs meet, they meet transversally. At each point, in the graphs, we can assign in a continuous manner normal vectors, instead of tangent vectors. At each point (a, b), where the graphs meet, we consider the matrix, whose rows are the normal vectors.

$$\begin{pmatrix} \dfrac{\partial g}{\partial X}(a,b), & \dfrac{\partial g}{\partial Y}(a,b) \\ \dfrac{\partial f}{\partial X}(a,b), & \dfrac{\partial f}{\partial Y}(a,b) \end{pmatrix}.$$

Since the graphs of $f = 0$ and $g = 0$ meet transversally at (a, b), the determinant of this matrix is not zero. For example, when $g = X^2 + Y^2 - 1$, the sign of this determinant measures, whether the "particle" tracing the graph of $f = 0$ is "entering" or "leaving" the interior of the circle at (a, b).

We will use these ideas in what follows, to show that if $m \subset (\mathbf{R}[X, Y])/(X^2 + Y^2 - 1) = A$ is a maximal ideal such that A/m is real, then m is not principal. This is well known, but the proof we give will motivate the definition of the "Euler class group" of a curve.

Proposition 3.1. *Let $A = (\mathbf{R}[X, Y])/(X^2+Y^2-1)$ be the coordinate ring of the real circle. Let (x_0, y_0) be a point on the circle and m, the corresponding maximal ideal of A. Then m is not principal.*

Proof. Suppose $m = (f(x,y))$. Then f generates m/m^2. In particular, the graph of $f(X,Y) = 0$ intersects the circle transversally at (x_0, y_0). Therefore, the normal to the graph of f and the normal to the circle at (x_0, y_0) form a basis of $\mathbf{R}^2$. Therefore, the determinant of the matrix

$$\begin{pmatrix} x_0, & y_0 \\ \dfrac{\partial f}{\partial X}(x_0, y_0), & \dfrac{\partial f}{\partial Y}(x_0, y_0) \end{pmatrix}$$

is equal to c, where c is different from 0.

Let $p : \mathbf{R} \to S^1$ be the covering projection sending t to $(\cos t, \sin t)$. Let $g : \mathbf{R} \to \mathbf{R}$ be defined by $g(t) = f(\cos t, \sin t)$. Assume that $(\cos t_0, \sin t_0) = (x_0, y_0)$. Then, we can verify easily using the chain rule that $g'(t)$ at $t = t_0$ is equal to c.

Assume now without loss of generality that $(x_0, y_0) \neq (0, 1)$. Then $g(0) = g(2\pi) \neq 0$. We have the following:

Fact. Let $g : \mathbf{R} \to \mathbf{R}$ be a continuously differentiable function such that $g(0) = g(2\pi) \neq 0$. Suppose $t_0 \in (0, 2\pi)$ such that $g(t_0) = 0$, $g'(t_0) \neq 0$. Then, there exists a $t_1 \in (0, 2\pi)$, $t_1 \neq t_0$, such that $g(t_1) = 0$.

Applying this fact to the function g defined as above we see that f vanishes on the circle at a point on the circle different from (x_0, y_0) showing that m is not principal. This proves the proposition.

More generally, we have the following.

Proposition 3.2. *Let $A = (\mathbf{R}[X,Y])/(X^2 + Y^2 - 1)$. Let $f \in A$ be such that the ideal $(f) = \cap_{i=1}^{r} m_i$ is the intersection of finitely many distinct real maximal ideals of A. Then r is even.*

Proof. The proof of this proposition is similar to the proof of Proposition 1. Rather than giving a formal proof, we explain the idea.

The graph of the function $f(X,Y) = 0$ can be thought of, as before, to be a path of a particle. This particle enters or leaves the interior of the circle at points on the circle corresponding to the maximal ideals

m_i. Whether it enters or leaves can be decided by looking at the sign of the determinant of the matrix

$$\begin{pmatrix} a_i, & b_i \\ \frac{\partial f}{\partial X}(a_i, b_i), & \frac{\partial f}{\partial Y}(a_i, b_i) \end{pmatrix}$$

where (a_i, b_i) is the point of S^1 corresponding to m_i. The proposition follows from the fact that if the particle enters the interior of the circle, then it also leaves the interior of the circle.

To see this, we note as we did earlier that the function $f : S^1 \to \mathbf{R}$ gives rise to $g : \mathbf{R} \to \mathbf{R}$ where $g(t) = f(\cos t, \sin t)$. Without loss of generality, we may assume as before that $g(0) = g(2\pi) \neq 0$.

The graph of function g can be thought of as the path of a particle. The particle crosses the X axis r times on $(0, 2\pi)$. Whether the particle is "entering" or "leaving" at these points can be decided by looking at the sign of the derivative of g at these points (that is, the points where g vanishes).

Let function g have finitely many zeroes $t_1, \dots, t_r$ on $[0, 2\pi]$. Since $g(0) = g(2\pi) \neq 0$ and $g'(t_i) \neq 0$ for every i, (we will show this shortly) we have $\sum_{i=1}^{r} \operatorname{sign}(g'(t_i)) = 0$, where $\operatorname{sign}(g'(t_i)) = 1$, if $g'(t_i)$ is positive and $\operatorname{sign}(g'(t_i)) = -1$ if $g'(t_i)$ is negative.

Let m_i correspond to the point (a_i, b_i) of S^1. The determinant of the matrix

$$\begin{pmatrix} a_i, & b_i \\ \frac{\partial f}{\partial X}(a_i, b_i), & \frac{\partial f}{\partial Y}(a_i, b_i) \end{pmatrix}$$

is u_i, where u_i is a unit of A/m_i, that is, u_i is a nonzero real number. As in Proposition 1 we see that $u_i = g'(t_i)$. Hence, $g'(t_i) \neq 0$. Further,

$$\sum_{i=1}^{r} \operatorname{sign}(u_i) = \sum_{i=1}^{r} \operatorname{sign}(g'(t_i)) = 0,$$

showing that r is even. This proves the proposition. $\square$

We would like to use the ideas in the proof of the proposition to define the "Euler class group" of $(\mathbf{R}[X, Y])/(X^2 + Y^2 - 1)$. To do this we associate to the polynomial function $\overline{f(X,Y)}$, defined on S^1 as in Proposition 3.2, the formal sum

$$\sum_{i=1}^{r}(m_i, u_i),$$

and we say that this sum "corresponds" to function f.

The question we ask is the following. Given a sum

$$\sum_{i=1}^{s}(m_i', u_i'),$$

where u_i' is a unit of A/m_i', does there exist a polynomial function h, defined on S^1, that is an element $h \in (\mathbf{R}[X,Y])/(X^2+Y^2-1)$ such that $(h) = \cap_{i=1}^{s} m_i'$ and the sum $\sum_{i=1}^{s}(m_i', u_i')$ corresponds to h?

Motivated by this question, we are led to the following tentative definition of the "Euler class group" of $A = (\mathbf{R}[X,Y])/(X^2+Y^2-1)$.

Definition. Let G be the free abelian group generated by pairs (m_i, u_i), where m_i is a real maximal ideal of $A = (\mathbf{R}[X,Y])/(X^2 + Y^2-1)$ and $u_i \in A/m_i$ is a unit.

Let H be the subgroup of G generated by

$$\sum_{i=1}^{s}(m_j', u_j'),$$

where $(h) = \cap_{i=1}^{s} m_j'$, ($m_j'$ distinct) and

$$\sum_{i=1}^{s}(m_j', u_j')$$

corresponds to h (in the sense explained above).

We define the Euler class group of A denoted by $E(A)$ to be the quotient G/H. In the above tentative definition, we have only considered real maximal ideals. We now give a definition which takes into account complex maximal ideals by defining the Euler class group of any regular one-dimensional affine domain. We begin with some considerations that motivate this definition. Let $\overline{f} \in (\mathbf{R}[X,Y])/(X^2+Y^2-1)$, be such that $\overline{f}$ generates m_i/m_i^2, where m_i is a real maximal ideal of $(\mathbf{R}[X,Y])/(X^2+Y^2-1)$ corresponding to the point (a_i,b_i) of S^1.

We have the following linear functions defined on $\mathbf{R}^2$.

1. $(\partial f/\partial X)(a_i,b_i)(X-a_i)+(\partial f/\partial Y)(a_i,b_i)(Y-b_i)$, which defines the tangent to the curve $f(X,Y)=0$ at the point (a_i,b_i).

2. $a_i(X-a_i)+b_i(Y-b_i)$, which defines the tangent to the circle at (a_i,b_i).

3. $-b_i(X-a_i)+a_i(Y-b_i)$, which defines the normal to the circle at (a_i,b_i).

The function, $\overline{-b_i(X-a_i)+a_i(Y-b_i)}$ is a basis for the one-dimensional vector space m_i/m_i^2. Let

$$\det\begin{pmatrix} a_i, & b_i \\ \dfrac{\partial f}{\partial X}(a_i,b_i), & \dfrac{\partial f}{\partial Y}(a_i,b_i)\end{pmatrix} = u_i.$$

We have

$$\det\begin{pmatrix} a_i, & b_i \\ -b_i, & a_i\end{pmatrix} = 1.$$

An easy computation then shows that

$$\overline{f} = \overline{u_i(-b_i(x-a_i)+a_i(y-b_i)} \text{ in } m_i/m_i^2.$$

<u>Now, we note that,</u> at each point $(a,b) \in S^1$, the normal $\overline{h'} = \overline{-b(x-a)+a(y-b)}$ generates m/m^2, where m is the corresponding real maximal ideal. Therefore, to a unit $\overline{u} \in A/m$, we can associate an element $\overline{uh'}$ of m/m^2. In view of this correspondence, the definition of the Euler class group of $A=(\mathbf{R}[X,Y])/(X^2+Y^2-1)$ can be rewritten as follows:

Definition. Let G be the free abelian group generated by the set of pairs (m,ω_m), where m is a real maximal ideal of A and ω_m is a generator of m/m^2.

Let H be the subgroup of G generated by elements of the kind

$$\sum_{j=1}^{s}(m_j,\omega_{m_j}), \quad (m_j \text{ distinct}),$$

where $g \in A$ is such that $(g) = \cap_{j=1}^{s} m_j$, and ω_{m_j} is the generator of m_j/m_j^2 given by g.

We define the Euler class group of A to be the quotient G/H.

Now we are ready for the final general definition of the Euler class group of any regular affine domain. (See [**1**, Section 4, Remark 4.6] for the definition of the Euler class groups of higher-dimensional varieties on which this definition is based.)

Definition 3.3. Let A be a regular affine domain over a field k with $\dim A = 1$. Let G be the free abelian group on the set of pairs (m, ω_m), where m is a maximal ideal of A and ω_m is a generator of m/m^2. Let H be the subgroup of G generated by elements of the kind $\sum_{j=1}^{s}(m_j, \omega_{m_j})$, where $(g) = \cap_{j=1}^{s} m_j$, (m_j distinct) and ω_{m_j} is the generator of m_j/m_j^2 given by g.

We define the Euler class group of A denoted by $E(A)$ to be the quotient G/H. $\square$

4. A heuristic argument. Let $f(X, Y) \in \mathbf{R}[X, Y]$ be a polynomial function such that the ideal $(f(X, Y), X^2 + Y^2 - 1) = \cap_{i=1}^{r} m_i$, where the m_i are distinct real maximal ideals of $\mathbf{R}[X, Y]$.

Let m_i correspond to the point (a_i, b_i) of S^1 and

$$\det \begin{pmatrix} a_i, & b_i \\ \dfrac{\partial f}{\partial X}(a_i, b_i), & \dfrac{\partial f}{\partial Y}(a_i, b_i) \end{pmatrix} = u_i.$$

Then, as in Proposition 2, we have

$$\sum_{i=1}^{r} \operatorname{sign}(u_i) = 0$$

We prove the following generalization.

Theorem 4.1. *Let $f(X, Y, Z)$, $g(Z, Y, Z) \in \mathbf{R}[X, Y, Z]$ be polynomial functions such that*

$$(f(X, Y, Z), g(X, Y, Z), X^2 + Y^2 + Z^2 - 1) = \cap_{i=1}^{r} m_i,$$

where m_i are real maximal ideals of $\mathbf{R}[X,Y,Z]$. *Let m_i correspond to the point (a_i, b_i, c_i) and*

$$\det\begin{pmatrix} a_i, & b_i, & c_i \\ \frac{\partial f}{\partial X}(a_i,b_i,c_i), & \frac{\partial f}{\partial Y}(a_i,b_i,c_i), & \frac{\partial f}{\partial Z}(a_i,b_i,c_i) \\ \frac{\partial g}{\partial X}(a_i,b_i,c_i), & \frac{\partial g}{\partial Y}(a_i,b_i,c_i), & \frac{\partial g}{\partial Z}(a_i,b_i,c_i) \end{pmatrix} = u_i.$$

Then $\sum_{i=1}^{r} \operatorname{sign}(u_i) = 0$.

We will first indicate an heuristic argument showing why this theorem should be true giving a rigorous argument later.

Proof. Assume that the common zeroes of $f(X,Y,Z)$ and $g(Z,Y,Z)$ define a smooth curve. The tangent plane to $f(X,Y,Z)$ at any real point (x_0, y_0, z_0) such that $f(x_0,y_0,z_0) = 0$ is given by:

$$\frac{\partial f}{\partial X}(x_0,y_0,z_0)(X-x_0) + \frac{\partial f}{\partial Y}(x_0,y_0,z_0)(Y-y_0) + \frac{\partial f}{\partial Z}(x_0,y_0,z_0)(Z-z_0) = 0.$$

Similarly, the tangent plane to $g(X,Y,Z)$ at any real point $(x_1, y_1 z_1)$ such that $g(x_1,y_1,z_1) = 0$ is given by:

$$\frac{\partial g}{\partial X}(x_1,y_1,z_1)(X-x_1) + \frac{\partial g}{\partial Y}(x_1,y_1,z_1)(Y-y_1) + \frac{\partial g}{\partial Z}(x_1,y_1,z_1)(Z-z_1) = 0.$$

Suppose that (x_2, y_2, z_2) is a common real zero of $f(X,Y,Z)$ and $g(X,Y,Z)$. Let

$$v_1 = \left(\frac{\partial f}{\partial X}(x_2,y_2,z_2),\ \frac{\partial g}{\partial Y}(x_2,y_2,z_2),\ \frac{\partial g}{\partial Z}(x_2,y_2,z_2)\right)$$

and

$$v_2 = \left(\frac{\partial g}{\partial X}(x_2,y_2,z_2),\ \frac{\partial g}{\partial Y}(x_2,y_2,z_2),\ \frac{\partial g}{\partial Z}(x_2,y_2,z_2)\right).$$

Let the cross product

$$v_1 \times v_2 = w_{(x_2,y_2,z_2)}.$$

Then $(x_2, y_2, z_2) + w_{(x_2,y_2,z_2)}$ is a tangent vector at (x_2, y_2, z_2) to the curve given by the common zeroes of $f(X,Y,Z), g(X,Y,Z))$, and the assignment sending each real point (x_2, y_2, z_2) belonging to the curve to $w_{(w_2,y_2,z_2)}$ is continuous.

Now, by assumption,

$$(f(X,Y,Z), g(X,Y,Z), X^2 + Y^2 + Z^2 - 1) = \bigcap_{i=1}^{r} m_i,$$

where m_i corresponds to the real point (a_i, b_i, c_i). A computation shows that $u_i = (a_i, b_i, c_i) \cdot w_{(a_i,b_i,c_i)}$, where $\cdot$ denotes the dot product of two vectors.

By assumption, the real points of the common zeroes of $f(X,Y,Z)$ and $g(X,Y,Z)$ form a smooth curve. This curve can be thought of as the path of a particle which enters and leaves the interior of the sphere. At each point (x_0, y_0, z_0) where the particle enters the interior of the sphere, the angle between $w_{(x_0,y_0,z_0)}$ and (x_0, y_0, z_0) is obtuse. At each point (x_1, y_1, z_1), where the particle leaves the interior of the sphere, the angle between $w_{(x_1,y_1,z_1)}$ and (x_1, y_1, z_1) is acute. Since

$$u_i = w_{(a_i,b_i,c_i)} \cdot (a_i, b_i, c_i),$$

we have $\sum_{i=1}^{r} \operatorname{sign}(u_i) = 0$. (That is, if the particle enters the interior of the sphere it also leaves the interior.)

5. The classification of one dimensional manifolds corresponding to smooth real algebraic varieties. The results of this section are devoted to making the heuristic argument given above rigorous. We shall use in this section the well-known classification of one-dimensional smooth real manifolds. We will present the details of this classification following [**5**, pages 264–267].

Let $f(X,Y,Z),\ g(X,Y,Z)$ be polynomial functions such that the ring

$$\frac{\mathbf{R}[X,Y,Z]}{f(X,Y,Z), g(X,Y,Z)} \text{ is regular.}$$

Assume also that polynomials f, g have at least one common real zero. By the Jacobian criterion we see that, at each common zero of $f(X,Y,Z), g(X,Y,Z)$, the determinant of at least one 2×2 minor of the matrix

$$\begin{pmatrix} \frac{\partial f}{\partial X}(a,b,c), & \frac{\partial f}{\partial Y}(a,b,c), & \frac{\partial f}{\partial Z}(a,b,c) \\ \frac{\partial g}{\partial X}(a,b,c), & \frac{\partial g}{\partial Y}(a,b,c), & \frac{\partial g}{\partial Z}(a,b,c) \end{pmatrix}$$

is nonzero.

By a one-dimensional manifold M (for the purposes of this discussion), we mean a path connected component in $\mathbf{R}^3$ of the set of real zeroes of $f(X,Y,Z)$, $g(X,Y,Z)$, where, $f(X,Y,Z)$, $g(X,Y,Z)$ are as above.

Definition 5.1. Let M be as above. Let $p = (p_1, p_2, p_3) \in M$. By the tangent space to M at p, we mean the set $(a,b,c) \in \mathbf{R}^3$ such that

$$\frac{\partial f}{\partial X}(p_1,p_2,p_3)(a-p_1) + \frac{\partial f}{\partial Y}(b-p_2) + \frac{\partial f}{\partial Z}(c-p_3) = 0$$

and

$$\frac{\partial g}{\partial X}(p_1,p_2,p_3)(a-p_1) + \frac{\partial g}{\partial Y}(b-p_2) + \frac{\partial g}{\partial Z}(c-p_3) = 0.$$

The tangent space to M at p is denoted by $T_p(M)$. By the tangent vector to M at p, we mean the affine subspace $p + T_p(M)$ of $\mathbf{R}^3$. (Note that if $(\mathbf{R}[X,Y,Z])/(f,g)$ is regular, by the Jacobian criterion $T_p(M)$ is one dimensional.) $\square$

Let M be a connected one-dimensional manifold as above and $p \in M$. Then, since $(\mathbf{R}[X,Y,Z])/(f(X,Y,Z), g(X,Y,Z))$ is regular, we can apply the implicit function to obtain a neighborhood $V \subset \mathbf{R}^3$ of p such that the following holds (after possibly relabeling coordinates):

(i) There exists a C^1 map from an open interval $\sigma : I \to \mathbf{R}^3$ sending t to $(t, g_1(t), g_2(t))$.

(ii) The image of σ is equal to $V \cap M$.

Definition 5.2. Let M be a connected one-dimensional manifold as above and $p \in M$. A regular local parametrization of M in a neighborhood of p is a C^1 map from an open interval I to $\mathbf{R}^3$ given by $\sigma(t) = (\sigma_1(t), \sigma_2(t), \sigma_3(t))$ satisfying:

(i) $\sigma(I) \subset M$.

(ii) $p \in \sigma(I)$.

(iii) for every $t \in I$, $\sigma'(t) = (\sigma_1'(t), \sigma_2'(t), \sigma_3'(t)) \neq (0,0,0)$. □

By the implicit function theorem, regular parametrizations exist in a neighborhood of any point $p \in M$.

Definition 5.3. Let M be as above. A regular local parametrization $\sigma : I \to M$ is said to have unit speed if $|\sigma'(t)| = 1$ for every $t \in I$.

Lemma 5.4. *Let M be as above. Then, regular local unit speed parametrizations exist in the neighborhood of any point $p \in M$.*

Proof. Let $\sigma : I \to M$ be a regular local parametrization of a neighborhood of $p \in M$. We can choose a closed interval $[a,b]$ contained in I such that $p = \sigma(t_0)$, $t_0 \in (a,b)$. Let $g : [a,b] \to \mathbf{R}$ be defined as follows:

$$g(t) = \int_a^t |\sigma'(t)|\, dt.$$

Then, since σ is regular, g maps $[a,b]$ in a one-to-one manner to an interval $[0,A]$. Let h be the inverse of g. Then σh maps $[0,A]$ to M. We claim that this is a unit speed parametrization. To see this, we compute

$$\begin{aligned}\frac{d}{ds}(\sigma(h(s)) &= \sigma'(h(s))h'(s) \\ &= \frac{1}{g'(h(s))}\sigma'(h(s)) \\ &= \frac{1}{|\sigma'(h(s))|}\sigma'(h(s)).\end{aligned}$$

Hence, σh has unit speed. This proves the lemma.

Lemma 5.5. *Let M be as above. Let $\sigma : I \to M$ be a regular local parametrization of M, where I is an open interval. Then $\sigma(I)$ is open in M.*

Proof. Let $t_0 \in I$. In the course of the proof we will shrink the interval I containing t_0 and assume that some further properties are satisfied.

Let $\sigma(t_0) = q = (q_1, q_2, q_3)$. Applying the implicit function theorem, we can choose a neighborhood V of q in $\mathbf{R}^3$ and a C^1 map, $\tau : I' \to M$ given by $\tau(t) = (t, \tau_2(t), \tau_3(t))$, such that the image of $\tau = V \cap M$. Let $\sigma = (\sigma_1, \sigma_2, \sigma_3)$. Since $\sigma(t_0) = (q_1, q_2, q_3)$ and $q_1 \in I'$, shrinking I, we may assume that $\sigma_1(I) \subset I'$. Further, we may also assume that $\sigma(I) \subset V \cap M$. Therefore, if $t \in I$, $\sigma(t) = (\sigma_1(t), \tau_2(\sigma_1(t), \tau_3(\sigma_1(t)))$. Now, since σ is regular, we see using the chain rule that $\sigma_1'(t_0) \neq 0$. Therefore, by the inverse function theorem, $\sigma_1(I)$ contains an interval I'' such that $q_1 \in I''$ and $I'' \subset I'$. Let $p_1 : \mathbf{R}^3 \to \mathbf{R}$ be the projection to the first coordinate. We choose a neighborhood W of q in $\mathbf{R}^3$ such that $p_1(W) \subset I''$. It is easy to see that $V \cap W \cap M \subset \sigma(I)$. Hence, $\sigma(I)$ is open in M. This proves the lemma. $\square$

Lemma 5.6. *Let M be as above. Let $\sigma : I \to M$, $\tau : J \to M$ be two regular local parametrizations such that $\sigma(t_0) = \tau(t_1)$. Then, there exists an open subinterval K of I containing t_0, and a C^1 function $\lambda : K \to J$, such that $\tau(\lambda(t)) = \sigma(t)$ for every $t \in K$.*

Proof. The idea of the proof is to set $\lambda = \tau^{-1}\sigma$. We give the details.

Let $\tau = (\tau_1, \tau_2, \tau_3)$. Since τ is regular, we may assume without loss of generality that $\tau_1'(t_1) \neq 0$. Shrinking J, we may assume that $\tau_1'(t) \neq 0$ for every $t \in J$. Therefore, as a consequence, we may assume that τ is one. We may assume further by the previous lemma, that there exists a neighborhood V of $\tau(t_1)$ in $\mathbf{R}^3$ such that $V \cap M \subset \tau(J)$. We can find an open subinterval K of I containing t_0 such that $\sigma(K) \subset V \cap M$. Let $p_1 : \mathbf{R}^3 \to \mathbf{R}$ be the projection onto the first coordinate. Since $\tau_1'(t) \neq 0$ for every $t \in J$, by the inverse function theorem, the map $p_1\tau$ is invertible with C^1 inverse μ. Let $\lambda = \mu p_1 \sigma$. Then λ satisfies the required property. This proves the lemma. $\square$

Lemma 5.7. *Let M be as above. Let $\phi, \psi : I \to M$ be regular local parametrizations which have unit speed at every point. If $\phi(t_0) = \psi(t_0)$ and $\phi'(t_0) = \psi'(t_0)$ for some point $t_0 \in I$, then $\phi(t) = \psi(t)$ for all $t \in I$.*

Proof. Since I is connected, it is enough to show the set of t, where $\phi(t) = \psi(t)$ is open. We show that $\phi(t) = \psi(t)$ in a neighborhood of t_0. By the previous lemma, there exists an interval K containing t_0 and a C^1 map $\lambda : K \to I$ such that $\phi = \psi \cdot \lambda$ and $\lambda(t_0) = t_0$. By the chain rule $\psi'(t) = \phi'(\lambda(t)\lambda'(t)$. Now since both ϕ' and ψ' have absolute value 1, it follows that λ' has absolute value 1 and since λ is C^1, λ' is identically equal to 1 or -1. Since $\phi'(t_0) = \psi'(t_0)$, $\lambda(t) = t$, near t_0. This proves the lemma. □

Lemma 5.8. *Let $\phi_1 : I_1 \to M$ and $\phi_2 : I_2 \to M$ be regular local unit speed parametrizations. If $\phi_1(I_1) \cap \phi_2(I_2)$ is not empty, then ϕ_1 has an extension $\phi : I \to M$ which is a regular local parametrization of unit speed satisfying $\phi(I) = \phi_1(I_1) \cup \phi_2(I_2)$.*

Proof. Suppose $\phi_1(t_1) = \phi_2(t_2)$. Set $\psi_2(t) = \phi_2(t - t_1 + t_2)$ on $I_2 + t_1 - t_2$.

It is clear that ψ_2 is a regular local unit speed parametrization. Further, $\psi_2(t_1) = \phi_2(t_2) = \phi_1(t_1)$. Since the tangent space to M at a is one-dimensional, there are just two possibilities for $\psi_2'(t_1)$: It is either $\phi_1'(t_1)$ or $-\phi_1'(t_1)$. In the first case, put $\psi = \psi_2$, and in the second case put $\psi(t) = \psi_2(2t_1 - t)$. Note that the map sending t to $2t_1 - t$ sends 0 to $2t_1$, $2t_1$ to 0, fixes t_1 and reverses orientation. By the previous lemma, $\psi = \phi_1$ on the interval where both are defined. Therefore, we can put $\phi = \phi_1$ on I_1, and $\phi = \psi$ on the interval, where ψ is defined. This function ϕ does the job. □

Lemma 5.9. *Let M be as above. Then there is a regular local unit speed parametrization $\phi : I \to M$ such that $\phi(I) = M$.*

Proof. Choose a point $a \in M$ and a unit tangent vector v to M at a. Let I be the union of all intervals J containing 0, such that there is a regular local parametric representation of unit speed $\phi_J : J \to M$

with $\phi_J(0) = a$ and $\phi'_J(0) = v$. Applying Lemma 5.4 (and reversing orientation if necessary), we see that ϕ_J certainly exists if J is small enough. If J and K are any two such intervals, then by Lemma 5.7, $\phi_J = \phi_K$ on $J \cap K$. Consequently, the ϕ_J determine a ϕ that is defined on the whole interval I. The function ϕ is a regular local unit speed parametrization of M. By construction, there is no extension of ϕ to a regular local unit speed parametrization defined on an interval larger than I. It follows by Lemma 5.5 that $\phi(I)$ is open in M. If we can show that $\phi(I)$ is closed in M, then by connectedness, we shall have $\phi(I) = M$. If $\phi(I)$ is not closed in M, let $b \in M$ be a point in the closure but not in $\phi(I)$ itself. Let $\psi : J \to M$ be a regular local unit speed parametrization such that $b \in \text{image}(\psi)$. Since $\psi(J)$ is open in M, we must have $\phi(I) \cap \psi(J)$ is not empty. Therefore, we can use the previous lemma to extend ϕ. Since this is impossible, it follows that the assumption that $\phi(I)$ is not closed in M does not hold up. This proves the lemma. $\square$

Lemma 5.10. *Let M be as above. Let $\phi : I \to M$ be a regular local unit speed parametrization. Suppose $t_0, t_1 \in I$ with $t_1 \neq t_0$, and $\phi(t_1) = \phi(t_0)$. Then $\phi'(t_1) = \phi'(t_0)$.*

Proof. Let $\phi(t_0) = \phi(t_1) = p$. Since $T_p(M)$ is one-dimensional, and $|\phi'(t_0)| = |\phi'(t_1)| = 1$, either $\phi'(t_0) = \phi'(t_1)$ or $\phi'(t_0) = -\phi'(t_1)$. Assume the second possibility holds. The restriction of ϕ to $[t_0, (t_0 + t_1)/2]$ is a function which corresponds to the path of a particle. The function sending t to $t_0 + t_1 - t$ takes the interval $[(t_0 + t_1)/2, t_1]$ to $[(t_0 + t_1)/2, t_0]$. The function $\psi : [(t_0 + t_1)/2, t_1] \to M$ given by $\psi(t) = \phi(t_0 + t_1 - t)$ corresponds to the path of a particle which retraces the above path backwards. We can extend ψ to a function $t_0 + t_1 - I \to M$ giving the same definition

$$(*) \qquad \psi(t) = \phi(t_0 + t_1 - t).$$

Note that $[t_0, t_1] \subset t_0 + t_1 - I$. Differentiating ψ at t_1, we get

$$\psi'(t_1) = -\phi'(t_0) = \phi'(t_1).$$

Further, $\psi(t_1) = \phi(t_0) = \phi(t_1)$. Hence, by Lemma 5.10, $\psi(t) = \phi(t)$ on $I \cap t_0 + t_1 - I$. Therefore,

$$\psi'\left(\frac{t_0 + t_1}{2}\right) = \phi'\left(\frac{t_0 + t_1}{2}\right).$$

However, differentiating $*$ at $(t_0+t_1)/2$, we get
$$\psi'\left(\frac{t_0+t_1}{2}\right) = -\phi'\left(\frac{t_0+t_1}{2}\right).$$
This is a contradiction and proves the lemma. $\square$

Lemma 5.11. *Let M be as above. Let $\phi : I \to M$ be a regular local unit speed parametrization. Suppose that ϕ is surjective. Let $t_1 > t_0$ and $\phi(t_1) = \phi(t_0)$. Then ϕ extends to a surjective regular unit speed parametrization $\psi : \mathbf{R} \to M$.*

Proof. Divide the real line into intervals $[t_0, t_1]$, $[t_1, t_1 + (t_1 - t_0)]$, etc., of length $t_1 - t_0$. Since $\phi(t_0) = \phi(t_1)$, we have $\phi'(t_0) = \phi'(t_1)$ (Lemma 5.10), and so by Lemma 5.7 we can extend ϕ to a C^1 function $\psi : \mathbf{R} \to M$ such that ψ is periodic of period $t_1 - t_0$. This proves the lemma. $\square$

Using a similar argument, we obtain

Lemma 5.12. *Let M be as above. Let $\phi : I \to M$ be a regular local unit speed parametrization such that ϕ is surjective. Assume that ϕ is not one-to-one. Then ϕ extends to a periodic C^1 function $\psi : \mathbf{R} \to M$ of period $\alpha > 0$, such that ψ is one-to-one on $(0, \alpha)$.*

We now prove Theorem 4.1, which we restate for convenience.

Theorem 5.13. *Let $f(X,Y,Z)$, $h(X,Y,Z) \in \mathbf{R}[X,Y,Z]$ be such that*
$$(f(X,Y,Z), h(X,Y,Z), X^2+Y^2+Z^2-1) = \cap_{i=1}^r m_i,$$
where m_i are real maximal ideals of $\mathbf{R}[X,Y,Z]$. Let m_i correspond to the point (a_i, b_i, c_i) and
$$\det\begin{pmatrix} a_i, & b_i, & c_i \\ \dfrac{\partial f}{\partial X}(a_i,b_i,c_i), & \dfrac{\partial f}{\partial Y}(a_i,b_i,c_i), & \dfrac{\partial f}{\partial Z}(a_i,b_i,c_i) \\ \dfrac{\partial h}{\partial X}(a_i,b_i,c_i), & \dfrac{\partial h}{\partial Y}(a_i,b_i,c_i), & \dfrac{\partial h}{\partial Z}(a_i,b_i,c_i) \end{pmatrix} = u_i.$$
Then $\sum_{i=1}^r \operatorname{sign}(u_i) = 0$.

Proof. The method of proof is based on the proofs of Propositions 3.1 and 3.2 and the heuristic argument we gave.

The ring $A = (\mathbf{R}[X,Y,Z])/(X^2+Y^2+Z^2-1)$ is regular. By Swan's Bertini theorem 2.3, we can choose $\lambda(x,y,z) \in A$ such that the element $f' = f + \lambda h$ satisfies the property that $A/(f')$ is regular.

Therefore, replacing f by $f + \lambda h$, we may assume that $A/(f)$ is regular. We note that this transformation does not change the value of the determinant above.

Since the ring $(\mathbf{R}[X,Y,Z])/(f(X,Y,Z), X^2+Y^2+Z^2-1)$ is regular, the common real zeroes of $(f(X,Y,Z), X^2+Y^2+Z^2-1)$ are the disjoint union $M_1 \cup M_2 \cup \cdots \cup M_{t-1} \cup M_t$ of path connected manifolds. We consider those maximal ideals say $m_1, \ldots, m_s$ which belong to a fixed connected component, say M_1. We will show that $\sum_{i=1}^{s} \operatorname{sign}(u_i) = 0$. If we do this for each connected component, the theorem will be proved.

Using the implicit function theorem, we see that each M_i is open in M.

By Lemma 5.9, there exists a surjective regular local unit speed parametrization $\sigma : I \to M_1$, where I is an open interval. We claim that σ is not one-to-one. Assume to the contrary that σ is one-to-one. Since I is an open interval, we have an open cover of I which has no finite subcover. Taking the image under σ, we obtain by Lemma 5.5, an open cover of M_1 which has no finite subcover. Now, since each M_i is open in M, we obtain an open cover of M which has no finite subcover. But M being a closed subset of S^2 is compact. This is a contradiction and hence σ is not one-to-one.

By Lemma 5.12, there exists a surjective C^1 map $\sigma : \mathbf{R} \to M_1$ such that σ is periodic of period $\alpha > 0$ and σ is one-to-one on $(0, \alpha)$.

Let (a,b,c) be any point of M_1. Let $v_1 = (a,b,c)$, $v_2 = ((\partial f/\partial X) \times (a,b,c), (\partial f/\partial Y)(a,b,c), (\partial f/\partial Z)(a,b,c))$ and $w_{(a,b,c)} = (v_1 \times v_2)/||v_1 \times v_2||$ where $\times$ denotes now the cross product of two vectors.

Let $\sigma(t) = (a,b,c)$ and $w'_{(a,b,c)} = (\sigma'(t))/||\sigma'(t)||$. The assignments sending $(a,b,c) \in M_1$ to $w_{(a,b,c)}$ and $w'_{(a,b,c)}$ are continuous. Note that $w_{(a,b,c)}$ and $w'_{(a,b,c)}$ are unit tangent vectors at each point $(a,b,c) \in M_1$. Since M_1 is connected, without loss of generality, we may assume that $w_{(a,b,c)} = w'_{(a,b,c)}$ for every $(a,b,c) \in M_1$.

Now, the polynomial function $h : M_1 \to \mathbf{R}$ gives a C^1 function $p = h\sigma : \mathbf{R} \to \mathbf{R}$. We may assume that $p(0) = p(\alpha) \neq 0$. We note that, since the function h has s zeroes on M_1, corresponding to the maximal ideals $m_1, \ldots, m_s$, the function p has s zeroes $t_1, \ldots, t_s$ on $(0, \alpha)$ corresponding to the maximal ideals $m_1, \ldots, m_s$. If we show that $p'(t_i) \neq 0$ for every i, then, since $p(0) = p(\alpha)$, as in Proposition 3.2,

$$\sum_{i=1}^{s} \operatorname{sign}(p'(t_i)) = 0.$$

Let $\sigma = (\sigma_1, \sigma_2, \sigma_3)$. Then $p(t) = h(\sigma_1(t), \sigma_2(t), \sigma_3(t))$. We have

$$\begin{aligned} p'(t) = {} & \frac{\partial h}{\partial X}(\sigma_1(t), \sigma_2(t), \sigma_3(t))\sigma_1'(t) + \frac{\partial h}{\partial Y}(\sigma_1(t), \sigma_2(t), \sigma_3(t))\sigma_2'(t) \\ & + \frac{\partial h}{\partial Z}(\sigma_1(t), \sigma_2(t), \sigma_3(t)) \cdot \sigma_3'(t). \end{aligned}$$

Hence,

$$\begin{aligned} p'(t_i) = {} & (\partial h/\partial X)(a_i, b_i, c_i)\sigma_1'(t_i) + (\partial h/\partial Y)(a_i, b_i, c_i)\sigma_2'(t_i) \\ & + \frac{\partial h}{\partial Z}(a_i, b_i, c_i)\sigma_3'(t_i). \end{aligned}$$

Note that $w_{(a_i,b_i,c_i)} = w'_{(a_i,b_i,c_i)}$ and $u_i \neq 0$. Now, a computation shows that $\operatorname{sign}(u_i) = \operatorname{sign}(p'(t_i))$. Hence, $p'(t_i) \neq 0$ and as before we see that

$$\sum_{i=1}^{s} \operatorname{sign}(p'(t_i)) = 0.$$

Further, $\sum_{i=1}^{s} \operatorname{sign}(u_i) = \sum_{i=1}^{s} \operatorname{sign}(p'(t_i)) = 0$. This proves the theorem. □

Remark 5.14. Let $A = (\mathbf{R}[X_1, \ldots, x_n])/(f_1, \ldots, f_{n-1})$ be a regular affine domain of dimension 1. Let $X = \operatorname{Spec} A$, and assume that $X(\mathbf{R})$ is not empty. Let M be a path connected component in $\mathbf{R}^n$ of $X(\mathbf{R})$. Then, the classification of one-dimensional manifolds given in this section also applies to M. □

6. Another proof of Theorem 4.1. In this section, we give another proof of Theorem 4.1. We first begin with some topological preliminaries.

Let us consider the circle S^1 which consists of real zeroes of $X^2 + Y^2 - 1$. We have map $\alpha : (-1,1) \to S^1$ given by $\alpha(t) = (t, \sqrt{1-t^2})$. We also have a map $\beta : (-1,1) \to S^1$ given by $\beta(t) = (t, -\sqrt{1-t^2})$. As a particle travels along the X axis from -1 to 1, its image under α travels in a clockwise direction on the circle, and its image under β travels in an anticlockwise direction. We formulate this observation using matrices.

Let $\alpha(t) = (\alpha_1(t), \alpha_2(t))$, where

$$\alpha_1(t) = t,\ \alpha_2(t) = \sqrt{1-t^2}.$$

The tangent vector to the curve $\alpha(t)$ at time t is given by the vector $(\alpha_1'(t), \alpha_2'(t))$,

$$(\,1, \quad -t/\sqrt{1-t^2}\,)\,.$$

We consider the matrix whose columns are given by the tangent vector and normal vector to the curve $\alpha(t)$ viz.,

$$\begin{pmatrix} 1, & t \\ \dfrac{-t}{\sqrt{1-t^2}} & \sqrt{1-t^2} \end{pmatrix}.$$

The determinant of this matrix is positive for each $t \in (-1,1)$.

Note that $\beta'(t) = (1,\ (t/\sqrt{1-t^2}))$. We see that the the matrix whose columns are given by the tangent and normal vector to the curve $\beta(t)$, viz.

$$\begin{pmatrix} 1, & t \\ \dfrac{t}{\sqrt{1-t^2}}, & -\sqrt{1-t^2} \end{pmatrix}$$

has negative determinant for each $t \in (-1,1)$.

We now consider the two-dimensional versions of the above statements.

Consider the disc $D = \{(a,b) \in \mathbf{R}^2 \mid a^2 + b^2 \leq 1\}$, and let

$$\sigma : D \longrightarrow S^2$$

be defined as

$$\sigma(u,v) = (u, v, \sqrt{1-u^2-v^2}),$$

and let $\tau : D \to S^2$ be defined as

$$\tau(u,v) = (u, v, -\sqrt{1-u^2-v^2}).$$

Note that, restricted to the open disc, σ and τ are C^∞ maps. We consider the 3×3 matrix β whose columns are $\sigma_u(u,v)$, $\sigma_v(u,v)$ and $(u, v, \sqrt{1-u^2-v^2})$, viz.

$$\begin{pmatrix} 1, & 0, & u \\ 0, & 1, & v \\ \dfrac{-u}{\sqrt{1-u^2-v^2}}, & \dfrac{-v}{\sqrt{1-u^2-v^2}}, & \sqrt{1-u^2-v^2} \end{pmatrix}.$$

This matrix has positive determinant for all (u,v) belonging to the open disc.

Similarly, we see that the matrix $\widetilde{\beta}$ whose columns are

$$\tau_u(u,v),\ \tau_v(u,v) \text{ and } (u, v, -\sqrt{1-u^2-v^2})$$

has negative determinant for all (u,v) in the open disc. (This matrix is explicitly written down in a following lemma).

We note that the vectors $\sigma_u(u,v)$, $\sigma_v(u,v)$, $\tau_u(u,v)$, $\tau_v(u,v)$ are tangent vectors to the two sphere S^2 and the vectors $(u, v, \sqrt{1-u^2-v^2})$, $(u, v, -\sqrt{1-u^2-v^2})$ are normal vectors to S^2. Now let $f, g : \mathbf{R}^3 \to \mathbf{R}$ be C^∞ functions. Let D^0 be the open disc $\{(a,b) \in \mathbf{R}^2 \mid a^2 + b^2 < 1\}$.

Define F, G from the open disc $D^0 \to \mathbf{R}$ as follows:

$$F(u,v) = f(u, v, \sqrt{1-u^2-v^2})$$
$$G(u,v) = g(u, v, \sqrt{1-u^2-v^2}).$$

Similarly, define F', G' from the open disc $D^0 \to \mathbf{R}$ as follows:

$$F'(u,v) = f(u, v, -\sqrt{1-u^2-v^2})$$
$$G'(u,v) = g(u, v, -\sqrt{1-u^2-v^2}).$$

With the notation as above, we have the following lemma whose proof follows from a computation which uses the chain rule.

Lemma 6.1. *Let* (u_0, v_0) *belong to the open disc of unit radius and* $(x_0, y_0, z_0) = (u_0, v_0, \sqrt{1 - u_0^2 - v_0^2})$*. Let*

$$\alpha = \begin{pmatrix} f_X(x_0, y_0, z_0), & f_Y(x_0, y_0, z_0), & f_Z(x_0, y_0, z_0) \\ g_X(x_0, y_0, z_0), & g_Y(x_0, y_0, z_0), & g_Z(x_0, y_0, z_0) \\ x_0, & y_0, & z_0 \end{pmatrix}$$

and

$$\beta = \begin{pmatrix} 1, & 0, & u_0 \\ 0, & 1, & v_0 \\ \dfrac{-u_0}{\sqrt{1 - u_0^2 - v_0^2}} & \dfrac{-v_0}{\sqrt{1 - u_0^2 - v_0^2}} & \sqrt{1 - u_0^2 - v_0^2} \end{pmatrix}.$$

Then

$$\alpha\beta = \begin{pmatrix} \dfrac{\partial F}{\partial u}(u_0, v_0), & \partial F \partial v(u_0, v_0), & 0 \\ \dfrac{\partial G}{\partial u}(u_0, v_0), & \dfrac{\partial G}{\partial v}(u_0, v_0), & 0 \\ *, & *, & 1 \end{pmatrix}.$$

Let

$$\widetilde{\alpha} = \begin{pmatrix} f_X(x_0, y_0, z_0), & f_Y(x_0, y_0, z_0), & f_Z(x_0, y_0, z_0) \\ g_X(x_0, y_0, z_0), & g_Y(x_0, y_0, z_0), & g_Z(x_0, y_0, z_0) \\ x_0, & y_0, & z_0 \end{pmatrix}$$

and

$$\widetilde{\beta} = \begin{pmatrix} 1, & 0, & u_0 \\ 0, & 1, & v_0 \\ \dfrac{u_0}{\sqrt{1 - u_0^2 - v_0^2}}, & \dfrac{v_0}{\sqrt{1 - u_0^2 - v_0^2}}, & -\sqrt{1 - u_0^2 - v_0^2} \end{pmatrix}.$$

Then

$$\widetilde{\alpha}\widetilde{\beta} = \begin{pmatrix} \dfrac{\partial F'}{\partial u}(u_0, v_0), & \partial F' \partial v(u_0, v_0), & 0 \\ \dfrac{\partial G'}{\partial u}(u_0, v_0), & \dfrac{\partial G'}{\partial v}(u_0, v_0), & 0 \\ *, & *, & 1 \end{pmatrix}.$$

In the next few lemmas, we will use x, y instead of X, Y to denote the coordinate functions in $\mathbf{R}^2$. The proofs of these lemmas are taken from [**4**, Appendix B and Appendix D].

Lemma 6.2. *Let $f : D \to \mathbf{R}$ be a C^∞ function where D is an open disc in $\mathbf{R}^2$ centered at $P = (a, b)$ in $\mathbf{R}^2$. Suppose $f(P) = 0$. Then there exist C^∞ functions $f_1, f_2 : D \to \mathbf{R}$ such that*

$$f(x, y) = (x - a)f_1(x, y) + (y - b)f_2(x, y).$$

Proof. Assume without loss of generality that $P = (0, 0)$. Let (x_0, y_0) be any point of the disc. We have

$$\begin{aligned} f(x_0, y_0) = f(x_0, y_0) - f(0, 0) &= \int_0^1 \frac{d}{dt}(f(tx_0, ty_0))\, dt \\ &= \int_0^1 \left[x_0 \frac{\partial f}{\partial x}(tx_0, ty_0) + y_0 \frac{\partial f}{\partial y}(tx_0, ty_0)\, dt \right] \\ &= x_0 \int_0^1 \frac{\partial f}{\partial x}(tx_0, ty_0)\, dt + y_0 \int \frac{\partial f}{\partial y}(tx_0, ty_0)\, dt. \end{aligned}$$

Set $f_1(x, y) = \int_0^1 (\partial f/\partial x)(tx, ty)\, dt$ and $f_2(x, y) = \int_0^1 (\partial f/\partial y)(tx, ty)\, dt$. Then the functions $f_1(x, y)$ and $f_2(x, y)$ satisfy the required properties. This proves the lemma. □

Now let $h(x, y), \tilde{h}(x, y) : D \to \mathbf{R}$ be C^∞ functions, where D is an open disc in $\mathbf{R}^2$ centered at the origin. Assume that $h(0, 0) = \tilde{h}(0, 0) = 0$. Then by Lemma 6.2 there exist C^∞ functions $h_1(x, y)$, $h_2(x, y)$, $\tilde{h}_1(x, y)$, $\tilde{h}_2(x, y) : D \to \mathbf{R}$ such that $h(x, y) = xh_1(x, y) + yh_2(x, y)$ and $\tilde{h}(x, y) = x\tilde{h}_1(x, y) + y\tilde{h}_2(x, y)$.

Lemma 6.3. *Let $h(x, y)$, $\tilde{h}(x, y) : D \to \mathbf{R}$ be C^∞ functions with the above properties. Assume in addition that*

$$(*) \qquad \det \begin{pmatrix} \dfrac{\partial h}{\partial x}(0, 0), & \dfrac{\partial h}{\partial y}(0, 0) \\ \dfrac{\partial \tilde{h}}{\partial x}(0, 0), & \dfrac{\partial \tilde{h}}{\partial y}(0, 0) \end{pmatrix} \neq 0.$$

Then there exists a closed disc $D' \subset D$ having the property that the image of the function

$$H : \partial D' \times I \longrightarrow \mathbf{R}^2$$

given by

$$H(x,y,t)=(xh_1(tx,ty)+yh_2(tx,ty),x\tilde{h}_1(tx,ty)+y\tilde{h}_2(tx,ty))$$

does not contain the origin. (*Here* $\partial D'$ *is the boundary circle of the closed disc* D', *and* I *denotes the interval* $[0,1]$).

Proof. By $(*)$, there exists a closed disc $D'\subset D$ such that the map $\lambda : D'\to \mathbf{R}^2$ given by

$$\lambda(x,y)=(h(x,y),\tilde{h}(x,y))$$

is injective.

We claim that the disc D' satisfies the property required by the lemma.

Let $(x,y)\in\partial D'$. Then, since $(x,y)\neq(0,0)$, by $*$ we have $H(x,y,0)\neq(0,0)$.

Let $t\neq 0$ and $(x,y)\in\partial D'$. We have $H(x,y,t)=(1/t)(h(tx,ty),\tilde{h}(tx,ty))$. Now, since λ is injective, $(h(tx,ty),\tilde{h}(tx,ty))\neq(h(0,0),\tilde{h}(0,0))=(0,0)$. Hence, $H(x,y,t)\neq(0,0)$. This proves the lemma. □

Before turning to the proof of the theorem we first recall the definition of the degree of a continuous map $\beta : S^1\to S^1$. We also prove a few lemmas on the degree of continuous maps which are needed in the proof. Let $p:\mathbf{R}\to S^1$ be the covering projection given by $p(t)=e^{2\pi it}$. Let $\alpha:[0,1]\to S^1$ be any continuous map such that $\alpha(0)=\alpha(1)$. To α, we can associate an integer as follows: Let $\delta:[0,1]\to\mathbf{R}$ be continuous and satisfy $p\delta=\alpha$. Then, we associate to α, the integer $\delta(1)-\delta(0)$. This integer does not depend on the choice of the lift δ.

Let $\beta : S^1\to S^1$ be any continuous map. We define $\alpha:[0,1]\to S^1$ by $\alpha(t)=\beta(e^{2\pi it})$, We can associate to α an integer N as above. We say that the degree of β is N.

Let $\beta' : S^1\to\mathbf{R}^2-\{(0,0)\}$ be any continuous map. We compose β' with the map $\lambda':\mathbf{R}^2-\{(0,0)\}\to S^1$ sending v to $v/||v||$. Then, we obtain a continuous map $\lambda'\beta' : S^1\to S^1$. We define the degree of β' to be the degree of the continuous map $\lambda'\beta'$.

We have the following easy lemma which we state without proof.

Lemma 6.4. *Let* $\beta_1, \beta_2 : S^1 \to \mathbf{R}^2 - \{(0,0)\}$ *be continuous maps which are homotopic. Then* $\operatorname{degree}(\beta_1) = \operatorname{degree}(\beta_2)$.

Now let $(a,b) \in \mathbf{R}$, and let D be the disc in $\mathbf{R}^2$ with center (a,b) and radius r. Then ∂D (the boundary of D) is homeomorphic to S^1. Let $\beta : \partial D \to \mathbf{R}^2 - \{(0,0)\}$ be any continuous map. Then, we can associate to β, as above, an integer degree(β). If $\beta_1, \beta_2 : \partial D \to \mathbf{R}^2 - \{(0,0)\}$ are two continuous maps which are homotopic, then β_1 and β_2 have the same degree.

Lemma 6.5. *Let* D *be an open disc in* $\mathbf{R}^2$ *centered at the origin. Let* $\lambda : D \to \mathbf{R}^2$ *be a* C^∞ *map such that* $\lambda(0,0) = 0$. *Assume that* $\lambda(x,y) = (h(x,y), \tilde{h}(x,y))$ *and*

$$\det \begin{pmatrix} \dfrac{\partial h}{\partial X}(0,0), & \dfrac{\partial h}{\partial Y}(0,0) \\ \dfrac{\partial \tilde{h}}{\partial X}(0,0), & \dfrac{\partial \tilde{h}}{\partial Y}(0,0) \end{pmatrix} = u \neq 0.$$

Then, there exists a closed disc $D' \subset D$ *centered at the origin satisfying the property that the image of the restriction of* $\lambda : \partial D' \to \mathbf{R}^2$ *does not contain the origin* $(0,0)$. *Further, the degree of the map* $\lambda : \partial D' \to \mathbf{R}^2 - (0,0)$ *is equal to* $\operatorname{sign}(u)$.

Proof. Let D' be chosen as in Lemma 6.3. Since degree is homotopy invariant, we may assume, by applying Lemma 6.3, that h and $\tilde{h}$ are linear functions. Now, we use the fact that any matrix of determinant 1 is a product of elementary matrices and the homotopy invariance of degree to prove the lemma in the case where h and $\tilde{h}$ are linear functions.

Lemma 6.6. *Let* D *be a closed disc in* $\mathbf{R}^2$. *Let* $\lambda : D \to \mathbf{R}^2$ *be continuous. Suppose* $\lambda(x,y) \neq (0,0)$ *for any* $(x,y) \in \partial D$. *Suppose further that* λ *is zero at only finitely many points* $(a_1,b_1), \ldots, (a_s,b_s) \in D$. *Choose closed discs* $D_i \subset D$ *centered at* (a_i,b_i) *and satisfying the property that* $\lambda(x,y) \neq (0,0)$ *for any* $(x,y) \in D_i$ *different from* (a_i,b_i). *Let degree* $\lambda : \partial D \to \mathbf{R}^2 - \{(0,0)\} = N$ *and degree* $\lambda_i : \partial D_i \to \mathbf{R}^2 - \{(0,0)\} = N_i$. *Then* $N = \sum_{i=1}^{s} N_i$.

We leave the proof of the above lemma to the reader. The proof is similar to the proof of Green's theorem in vector calculus for multiply connected regions.

Now we give another proof of Theorem 4.1.

Theorem 6.7. *Let $f(X,Y,Z)$, $g(X,Y,Z) \in \mathbf{R}[X,Y,Z]$ be such that*

$$(f(X,Y,Z), g(X,Y,Z), X^2+Y^2+Z^2-1) = \bigcap_{i=1}^{r} m_i$$

where m_i are real maximal ideals of $\mathbf{R}[X,Y,Z]$. Let m_i correspond to the point (a_i,b_i,c_i) of S^2 and

$$\det\begin{pmatrix} a_i, & b_i, & c_i \\ f_X(a_i,b_i,c_i), & f_Y(a_i,b_i,c_i), & f_Z(a_i,b_i,c_i) \\ g_X(a_i,b_i,c_i), & g_Y(a_i,b_i,c_i), & g_Z(a_i,b_i,c_i) \end{pmatrix} = u_i.$$

Then $\sum_{i=1}^{r} \operatorname{sign}(u_i) = 0$.

Proof. We assume, without loss of generality by performing an orthogonal transformation if necessary, that none of the points (a_i,b_i,c_i) lie on the equator of the two-sphere S^2. (We note that an orthogonal transformation induces on an automorphism of $(\mathbf{R}[X,Y,Z])/(X^2+Y^2+Z^2-1)$ and, using the chain rule, it follows that if the theorem follows after making an orthogonal transformation then it follows anyway.)

Let D be the disc $\{(u,v) \in \mathbf{R}^2 \mid u^2+v^2 \le 1\}$, and let $F, G, F', G' : D \to \mathbf{R}$ be defined as follows:

$$F(u,v) = f(u,v,\sqrt{1-u^2-v^2}), \quad G(u,v) = g(u,v,\sqrt{1-u^2-v^2}),$$
$$F'(u,v) = f(u,v,-\sqrt{1-u^2-v^2}), \quad G'(u,v) = g(u,v,-\sqrt{1-u^2-v^2}).$$

We consider the maps $\lambda, \lambda' : D \to \mathbf{R}^2$ defined as

$$\lambda(u,v) = (F(u,v),\ G(u,v))$$
$$\lambda'(u,v) = (F'(u,v),\ G'(u,v))).$$

Note that λ, λ' coincide on ∂D (the boundary circle of D).

By Lemma 6.1, if (a_i, b_i, c_i) belongs to the upper hemisphere, the sign of

$$\det \begin{pmatrix} f_X(a_i,b_i,c_i), & f_Y(a_i,b_i,c_i), & f_Z(a_i,b_i,c_i) \\ g_X(a_i,b_i,c_i), & g_Y(a_i,b_i,c_i), & g_Z(a_i,b_i,c_i) \\ a_i, & b_i, & c_i \end{pmatrix}$$

is the same as the sign of

$$\det \begin{pmatrix} \frac{\partial F}{\partial u}(a_i,b_i), & \frac{\partial F}{\partial v}(a_i,b_i) \\ \frac{\partial G}{\partial u}(a_i,b_i), & \frac{\partial G}{\partial v}(a_i,b_i) \end{pmatrix}.$$

By the same lemma, if (a_i, b_i, c_i) belongs to the southern hemisphere, the sign of

$$\det \begin{pmatrix} f_X(a_i,b_i,c_i), & f_Y(a_i,b_i,c_i), & f_Z(a_i,b_i,c_i) \\ g_X(a_i,b_i,c_i), & g_Y(a_i,b_i,c_i), & g_Z(a_i,b_i,c_i) \\ a_i, & b_i, & c_i \end{pmatrix}$$

is opposite to the sign of

$$\det \begin{pmatrix} \frac{\partial F'}{\partial u}(a_i,b_i), & \frac{\partial F'}{\partial v}(a_i,b_i) \\ \frac{\partial G'}{\partial u}(a_i,b_i), & \frac{\partial G'}{\partial v}(a_i,b_i) \end{pmatrix}.$$

Now, since the restrictions of λ and λ' to ∂D are equal, the theorem follows from Lemmas 6.5 and 6.6.

7. On the Euler class group of smooth real curves and surfaces. In this section, we use surjective homomorphisms from the Euler class groups of real curves and surfaces to free abelian groups. We use the classification of one manifolds given in Section 5.

We begin by recalling the definition of the Euler class group of a curve.

Definition 7.1. Let A be a regular affine domain over a field k with $\dim A = 1$. Let G be the free abelian group on the set of pairs (m, ω_m), where m is a maximal ideal of A and ω_m is a generator of m/m^2.

Let H be the subgroup of G generated by $\sum_{i=1}^{s}(m_i, \omega_{m_i})$ where $(f) = \cap_{i=1}^{s} m_i$, (m_i distinct) and ω_{m_i} is the generator of m_i/m_i^2 given by f. We recall that the Euler class group $E(A)$ of A is defined to be the quotient G/H. $\square$

Now let $A = (\mathbf{R}[X,Y])/(X^2+Y^2-1)$. We define a homomorphism $\phi : E(A) \to \mathbf{Z}$ as follows:

Let m be a maximal ideal of A, and let ω_m be a generator of m/m^2. If m is complex, we set $\phi((m,\omega_m)) = 0$. If m is real and corresponds to the point $(a,b) \in S^1$, and ω_m is the generator of m/m^2 given by $f(x,y)$, we set $\phi((m,\omega_m)) = \operatorname{sign}(u)$, where

$$u = \det \begin{pmatrix} \dfrac{\partial f}{\partial X}(a,b), & \dfrac{\partial f}{\partial Y}(a,b) \\ a, & b \end{pmatrix}.$$

Thus, we obtain a homomorphism $\phi : G \to \mathbf{Z}$, (where G is as in the definition of the Euler class group). Imitating the proof of Proposition 3.2, we see that $\phi(H) = 0$. Thus, we obtain a homomorphism $\phi : E(A) \to \mathbf{Z}$.

More generally, let $A = (\mathbf{R}[X,Y))/(g(X,Y))$ be a regular affine domain of dimension 1. Let $X = \operatorname{Spec} A$, and let $X(\mathbf{R})$ be the set of real points of X with the topology on $X(\mathbf{R})$ induced from $\mathbf{R}^2$. Suppose $X(\mathbf{R})$ has t compact path connected components $M_1, M_2, \ldots, M_t$. Let $E(A) = G/H$, where G and H are defined as above.

We define a homomorphism $\phi : G \to \mathbf{Z}^t$ as follows:

Let m be a maximal ideal of A and ω_m be a generator of m/m^2. If m is a complex maximal ideal, we set $\phi((m,\omega_m)) = 0$. If m is a real maximal ideal such that the associated real point belongs to a noncompact connected component of $X(\mathbf{R})$, we set $\phi((m,\omega_m)) = 0$. Let m be a real maximal ideal corresponding to a point (a,b) belonging to a compact connected component M_i of $X(\mathbf{R})$, and let ω_m be the generator of m/m^2 given by f. Let

$$\det \begin{pmatrix} \dfrac{\partial f}{\partial X}(a,b), & \dfrac{\partial f}{\partial Y}(a,b) \\ \dfrac{\partial g}{\partial X}(a,b), & \dfrac{\partial g}{\partial Y}(a,b) \end{pmatrix} = u.$$

We set $\phi((m,\omega_m)) = (0,0,\dots,0,1,0,\dots 0)$, (where 1 is in the ith place) if u is positive. We set $\phi((m,\omega_m)) = (0,0,\dots,0,-1,0\dots,0)$ (where -1 is in the ith place) if u is negative. Thus, we obtain a homomorphism $\phi : G \to \mathbf{Z}^t$, (where G is as in the definition of $E(A)$, i.e., $E(A) = G/H$). Imitating the arguments in Section 5 (see 5.14), we see that there exists a C^1 diffeomorphism $\phi_i : S^1 \to M_i$ for every i. Now, mimicking the arguments used in the proofs of Theorem 5.13 and Proposition 3.2, we see that $\phi(H) = 0$. Thus, we obtain the following theorem which is a one-dimensional version of [**2**, Proposition 4.12].

Theorem 7.2. *Let $A = (\mathbf{R}[X,Y])/(g(X,Y))$ be a regular affine domain of dimension 1. Let $X = \operatorname{Spec} A$, and suppose that $X(\mathbf{R})$ has t compact connected components. Then, there exists a surjective homomorphism $\phi : E(A) \to \mathbf{Z}^t$.*

We next turn our attention to the Euler class group of surfaces. We begin by giving a tentative definition of the Euler class group of the coordinate ring of the two sphere S^2. This will motivate the general definition of the Euler class group of a surface. We have used similar ideas earlier to define the Euler class group of a curve. So we will not give all the details.

Let $A = (\mathbf{R}[X,Y,Z])/(X^2+Y^2+Z^2-1)$. Suppose $f, g \in A$ are such that $(f,g) = \cap_{i=1}^r m_i$, where m_i are distinct real maximal ideals of A. Let m_i correspond to the point (a_i,b_i,c_i) of S^2.

Let

$$u_i = \det \begin{pmatrix} \dfrac{\partial f}{\partial X}(a_i,b_i,c_i), & \dfrac{\partial f}{\partial Y}(a_i,b_i,c_i), & \dfrac{\partial f}{\partial Z}(a_i,b_i,c_i) \\ \dfrac{\partial g}{\partial X}(a_i,b_i,c_i), & \dfrac{\partial g}{\partial Y}(a_i,b_i,c_i), & \dfrac{\partial g}{\partial Z}(a_i,b_i,c_i) \\ a_i, & b_i, & c_i \end{pmatrix}.$$

Then u_i is a nonzero real number, that is, u_i is a unit of A/m_i.

Definition 7.2. Let $A = (\mathbf{R}[X,Y,Z])/(X^2+Y^2+Z^2-1)$. Let G be the free abelian group on the set of pairs (m,u), where m is a real maximal ideal of A and $u \in A/m$ is a unit.

Let $(f,g) = \cap_{i=1}^r m_i$ be as above, and let $u_i \in A/m_i$ be the unit constructed above. To (f,g), we associate the element $\sum_{i=1}^r (m_i,u_i)$ of G.

Let H be the subgroup of G generated by the elements which are constructed in the above manner. We define the Euler class group of A to be the quotient G/H.

We would like to modify this tentative definition and give a general definition of the Euler class group of the coordinate ring of a smooth affine surface. The following considerations, motivate this definition. Let (a, b, c) be any point of S^2. Choose vectors $v_1 = (\lambda_{11}, \lambda_{12}, \lambda_{13})$ and $v_2 = (\lambda_{21}, \lambda_{22}, \lambda_{23})$ such that the matrix

$$\begin{pmatrix} \lambda_{11}, & \lambda_{12}, & \lambda_{13} \\ \lambda_{21}, & \lambda_{22}, & \lambda_{23} \\ a, & b, & c \end{pmatrix}$$

has determinant 1.

Let

$$f_1 = \lambda_{11}(X-a) + \lambda_{12}(Y-b) + \lambda_{13}(Z-c)$$
$$f_2 = \lambda_{21}(X-a) + \lambda_{22}(Y-b) + \lambda_{23}(Z-c).$$

Then, f_1, f_2 generate m/m^2, where m is the maximal ideal of $A = (\mathbf{R}[X, Y, Z])/(X^2 + Y^2 + Z^2 - 1)$ corresponding to the point (a, b, c). Thus, to each real maximal ideal m of A, we can assign an oriented basis of m/m^2, and hence an element $\omega_m = \bar{f}_1 \wedge \bar{f}_2$ of $\wedge^2 m/m^2$. Let $f, g \in A$ be such that

$$(f, g) = \bigcap_{i=1}^{r} m_i,$$

and let $\sum (m_i, u_i)$ be the element of G associated to (f, g). Then, a computation shows that

$$\bar{f} \bigwedge \bar{g} = u_i \omega_{m_i} \text{ in } \bigwedge^2 m_i/m_i^2.$$

Now, we use the above considerations to modify the tentative definition of the Euler class group of the 2-sphere and give a general definition of the Euler class group of a surface. (See [**1**, Section 4, Remark 4.6].)

Definition 7.3. Let A be a regular affine domain over a field k with $\dim A = 2$. Let G be the free abelian group on the set of pairs (m, ω_m), where m is a maximal ideal of A and ω_m is a generator of $\wedge^2 m/m^2$.

Let H be the subgroup of G generated by elements of the kind $\sum_{i=1}^{r}(m_i, \omega_{m_i})$, ($m_i$ distinct) where $\cap_{i=1}^{r} m_i = (f, g)$, and ω_{m_i} is the generator of $\wedge^2 m_i/m_i^2$ given by $\bar{f} \wedge \bar{g}$.

We define the Euler class group of A denoted by $E(A)$ to be the quotient G/H. Let $A = (\mathbf{R}[X, Y, Z])/(X^2 + Y^2 + Z^2 - 1)$. We define a homomorphism $\phi : E(A) \to \mathbf{Z}$ as follows:

Let m be a maximal ideal of A, and let ω_m be a generator of $\wedge^2 m/m^2$. If m is complex, we set $\phi((m, \omega_m)) = 0$. Let m be a real maximal ideal of A corresponding to the point $(a, b, c) \in S^2$, and let ω_m be a generator of $\wedge^2/m/m^2$ given by $\bar{f} \wedge \bar{g}$, where f, g generate m/m^2. Let

$$u = \det \begin{pmatrix} \dfrac{\partial f}{\partial X}(a,b,c), & \dfrac{\partial f}{\partial Y}(a,b,c), & \dfrac{\partial f}{\partial Z}(a,b,c) \\ \dfrac{\partial g}{\partial X}(a,b,c), & \dfrac{\partial g}{\partial Y}(a,b,c), & \dfrac{\partial g}{\partial Z}(a,b,c) \\ a, & b, & c \end{pmatrix}.$$

We set $\phi((m, \omega_m)) = \operatorname{sign}(u) \in \mathbf{Z}$. Thus, we obtain a homomorphism $\phi : G \to \mathbf{Z}$, (where G is as in the definition of the Euler class group). By Theorem 5.13, we have $\phi(H) = 0$. (Note that in Theorem 5.13, there is no mention of complex maximal ideals but the same argument applies.) Thus, we obtain a surjection $\phi : E(A) \to \mathbf{Z}$, where $A = (\mathbf{R}[X, Y, Z])/(X^2 + Y^2 + Z^2 - 1)$. It can be shown that ϕ is an isomorphism. (See [**2**, Remark 5.8].)

More generally, let $A = (\mathbf{R}[X, Y, Z])/(h(X, Y, Z))$ be a regular affine domain of dimension 2. Let $X = \operatorname{Spec} A$ and assume that $X(\mathbf{R})$ has t compact path connected components $M_1, \ldots, M_t$. Let $E(A) = G/H$, where G and H are defined as above. We define a homomorphism $\phi : G \to \mathbf{Z}^t$ as follows:

Let m be a maximal ideal of A and ω_m be a generator of $\wedge^2 m/m^2$. If m is a complex maximal ideal, we set $\phi((m, \omega_m)) = 0$. If m is a real maximal ideal such that the associated real point belongs to a noncompact connected component of $X(\mathbf{R})$, of A we set $\phi((m, \omega_m)) = 0$.

Let m be a real maximal ideal corresponding to a point (a, b, c) belonging to a compact connected component M_i of $X(\mathbf{R})$, and let ω_m be a generator of $\wedge^2 m/m^2$ given by $\bar{f} \wedge \bar{g}$, where $f, g \in m$ generate

m/m^2. Let

$$u = \det \begin{pmatrix} \frac{\partial f}{\partial X}(a,b,c), & \frac{\partial f}{\partial Y}(a,b,c), & \frac{\partial f}{\partial Z}(a,b,c) \\ \frac{\partial g}{\partial X}(a,b,c), & \frac{\partial g}{\partial Y}(a,b,c), & \frac{\partial g}{\partial Z}(a,b,c) \\ \frac{\partial h}{\partial X}(a,b,c), & \frac{\partial h}{\partial Y}(a,b,c), & \frac{\partial h}{\partial Z}(a,b,c) \end{pmatrix}.$$

We set $\phi((m,\omega_m)) = (0,0,\dots,0,1,0,\dots,0)$, (where the 1 is in the ith place) if u is positive. We set $\phi((m,\omega_m)) = (0,0,\dots,0,-1,0,\dots,0)$, (where the -1 is in the ith place) if u is negative. Thus, we obtain a homomorphism $\phi : G \to \mathbf{Z}^t$. Using Corollary 2.3 and an argument similar to the proof of Theorem 5.13, we see that $\phi(H) = 0$. Thus, we obtain the following.

Theorem 7.4. *Let $A = (\mathbf{R}[X,Y,Z])/(h(X,Y,Z))$ be a regular affine domain of dimension 2. Let $X = \operatorname{Spec} A$, and suppose that $X(\mathbf{R})$ has t compact connected components. Then there exists a surjective homomorphism $\phi : E(A) \to \mathbf{Z}^t$.*

Now we state the general definition of the Euler class group of an n-dimensional variety given in [**1**, Section 4, Remark 4.6].

Definition 7.5. Let A be a regular affine domain over an infinite field k with $\dim A = n$. Let G be the free abelian group on the set of pairs (m,ω_m), where m is a maximal ideal of A and ω_m is a generator of $\wedge^n m/m^2$. Let H be the subgroup of G generated by elements of the kind $\sum_{i=1}^r (m_i,\omega_{m_i})$ (m_i distinct), where $\cap_{i=1}^r m_i = (f_1,\dots,f_n)$ and ω_{m_i} is the generator of $\wedge^n m_i/m_i^2$ given by $\bar{f}_1 \wedge \bar{f}_2 \wedge \cdots \wedge \bar{f}_n$. We define the Euler class group of A denoted by $E(A)$ to be the quotient G/H.

Using Corollary 2.3 and the method of proof of Theorem 5.13, we can prove the following theorem which is a special case of [**2**, Proposition 4.12].

Theorem 7.6. *Let $A = (\mathbf{R}[X_1,X_2,\dots,X_{n+1}])/(h(X_1,\dots,X_{n+1}))$ be a regular affine domain over $\mathbf{R}$ of dimension n. Let $X = \operatorname{Spec} A$, and assume that $X(\mathbf{R})$ has t compact connected components. Then there exists a surjective homomorphism $\phi : E(A) \to \mathbf{Z}^t$.*

8. The Euler class of an oriented projective module. In this section, we relate the Euler class of a rank 2 oriented projective module over the coordinate ring of the two sphere to the "degree" of a certain continuous map $S^1 \to SL_2(\mathbf{R})$.

We begin by recalling how the degree of a continuous map $\beta : S^1 \to \mathbf{R}^2 - \{(0,0)\}$ is defined. We compose β with the map $\lambda : \mathbf{R}^2 - \{(0,0)\} \to S^1$ sending $v \in \mathbf{R}^2 - \{(0,0)\}$ to $v/||v||$, thus obtaining a map $\lambda\beta : S^1 \to S^1$. We recall that the degree(β) is defined to be the degree($\lambda\beta$).

Let $\alpha : S^1 \to SL_2(\mathbf{R})$ be continuous. We consider the map $\lambda : SL_2(\mathbf{R}) \to \mathbf{R}^2 - \{(0,0)\}$, sending any element of $SL_2(\mathbf{R})$ to its first row. Composing α with λ we obtain a continuous map $\lambda\alpha : S^1 \to \mathbf{R}^2 - \{(0,0)\}$. We define the degree of α to be degree($\lambda\alpha$).

We recall that the Gram-Schmidt orthogonalization process gives us a continuous retraction $\mu : SL_2(\mathbf{R}) \to SO_2(\mathbf{R})$. We recall how this retraction is constructed.

Any matrix belonging to $SL_2(\mathbf{R})$ consists of a pair of linearly independent column vectors (v, w). We first choose $\lambda \in \mathbf{R}$ so that the dot product $w \cdot (v - \lambda w) = 0$. Let $v_1 = v - \lambda w$. We set $\mu((v, w)) = (v_1/||v_1||, w/||w||)$. We record the following lemma which follows easily from the Gram-Schmidt process.

Lemma 8.1. *There exists a continuous map $H : SL_2(\mathbf{R}) \times I \to GL_2(\mathbf{R})$ such that $H(\sigma, 0) = \sigma$, and $H(\sigma, 1) = \mu(\sigma)$ for every $\sigma \in SL_2(\mathbf{R})$.*

Lemma 8.2. *Let $\mu_1 = (f_1, f_2) : S^1 \to \mathbf{R}^2 - \{(0,0)\}$ and $\mu_2 = (g_1, g_2) : S^1 \to \mathbf{R}^2 - \{(0,0)\}$ be continuous. Suppose there exists a continuous map $\sigma : S^1 \to SL_2(\mathbf{R})$ such that*

$$\sigma(a,b)\begin{pmatrix} f_1(a,b) \\ f_2(a,b) \end{pmatrix} = \begin{pmatrix} g_1(a,b) \\ g_2(a,b) \end{pmatrix}$$

for all $(a, b) \in S^1$. Then $\deg(\mu_1) - \deg(\mu_2) = \deg(\sigma)$.

Proof. We may by the previous lemma replace σ by a map homotopic to σ and assume that the image of σ is contained in $SO_2(\mathbf{R})$. We may also replace μ_1 by a homotopic map and assume that the image of μ_1

is contained in S^1. Now, note that the following identity holds

$$\begin{pmatrix} \cos\theta, & \sin\theta \\ -\sin\theta, & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} = \begin{pmatrix} \cos(\phi-\theta) \\ \sin(\phi-\theta) \end{pmatrix}.$$

The lemma now follows from the fact that if $\beta_1, \beta_2 : S^1 \to S^1$ are continuous maps and if $\beta_1 \cdot \beta_2 : S^1 \to S^1$ sends $z \in S^1$ to $\beta_1(z) \cdot \beta_2(z)$, then $\deg(\beta_1 \cdot \beta_2) = \deg(\beta_1) + \deg(\beta_2)$. $\square$

We next turn to the definition of the Euler class of an oriented projective module given in [**1**, Section 4].

Before doing this, we recall the definition of the Euler class group $E(A)$ of A.

Definition 8.3. Let A be a regular affine domain over an infinite perfect field. Let G be the free abelian group on the set of pairs (m, ω_m), where m is a maximal ideal of A and ω_m is an orientation of m (that is a generator of $\wedge^n m/m^2$).

Let H be the subgroup of G generated by $\sum_{i=1}^r (m_i, \omega_{m_i})$, where $(f_1, \dots, f_n) = \cap_{i=1}^r m_i$ (m_i distinct) and ω_{m_i} is the orientation of m_i given by $\bar{f}_1 \wedge \cdots \wedge \bar{f}_n$. We recall that the Euler class group $E(A)$ of A is defined to be the quotient G/H. $\square$

Let P be a projective A-module of rank n and trivial determinant. A generator of $\wedge^n P$ (or equivalently an isomorphism $A \xrightarrow{\sim} \wedge^n P$) is called an orientation of P. An oriented projective module P of rank n is a projective module P of rank n and trivial determinant together with an orientation of P.

Let P be an oriented projective A-module of rank n, and let χ be an orientation of P (that is, χ is a generator of $\wedge^n P$). To the pair (P, χ), we assign an element of $E(A)$, as follows: Let $\alpha : P \to \cap_{i=1}^s m_i$ be a generic surjection (where the m_i are distinct maximal ideals of A). (See subsection 2.5). Then, we have an isomorphism

$$\wedge^n(\overline{\alpha}) : \wedge^n P/m_i P \xrightarrow{\sim} \wedge^n m_i/m_i^2.$$

The image of the orientation $\overline{\chi}$ under the isomorphism $\wedge^n(\overline{\alpha})$ is an orientation ω_{m_i} of m_i (that is, a generator of $\wedge^n m_i/m_i^2$).

We assign to the pair (P, χ) the element $\sum_{i=1}^{s}(m_i, \omega_{m_i})$ of $E(A)$. The "oriented cycle" $\sum_{i=1}^{s}(m_i, \omega_{m_i})$ is called the Euler class of the oriented projective module (P, χ). The Euler class of an oriented projective module (P, χ) does not depend upon the choice of the generic surjection α and is denoted by $e(P, \chi)$. (See [**1**] for details and proofs.)

Now, let $A = (\mathbf{R}[X, Y, Z])/(X^2 + Y^2 + Z^2 - 1)$, and let P be a projective module of rank 2. Since $\operatorname{Pic} A$ is trivial, the determinant of P is trivial and P can be oriented. Let $\chi : A \xrightarrow{\sim} \wedge^2 P$ be an isomorphism defining an orientation of P.

Since vector bundles over the affine plane are trivial, one can easily show, using stereographic projection, that the projective A_{z-1} module P_{z-1} and the projective A_{z+1} module P_{z+1} are both free of rank 2.

We choose isomorphism $\beta_1 : A_{z+1}^2 \xrightarrow{\sim} P_{z+1}$ and $\beta_2 : A_{z-1}^2 \xrightarrow{\sim} P_{z-1}$ such that

$$\wedge^2 \beta_1 = \chi \text{ and } \wedge^2 \beta_2 = \chi.$$

We would like to describe the Euler class of the oriented projective module (P, χ) in terms of this data.

Let $\alpha : P \to \cap_{i=1}^{s} m_i$ be a surjection where m_i are (distinct) maximal ideals of A. Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Let

$$\begin{aligned} \alpha\beta_1(e_1) &= f_1, \qquad & \alpha\beta_1(e_2) &= f_2 \\ \alpha\beta_2(e_1) &= f_2, \qquad & \alpha\beta_2(e_2) &= g_2. \end{aligned}$$

Let us consider the maximal ideals $m_1, \dots, m_s$. If $z + 1 \notin m_i$, then $\bar{f}_1 \wedge \bar{f}_2$ is an orientation ω_{m_i} of m_i. If $z - 1 \notin m_j$, then $\bar{g}_1 \wedge \bar{g}_2$ is an orientation ω_{m_j} of m_j. If $(z-1)(z+1) \notin m_k$, then since $\wedge^2 \beta_1 = \chi$ and $\wedge^2 \beta_2 = \chi$, we have the orientations of m_i given by $\bar{f}_1 \wedge \bar{f}_2$ and $\bar{g}_1 \wedge \bar{g}_2$ are the same. One can verify that $e(P, \chi) = \sum_{i=1}^{s}(m_i, \omega_{m_i})$.

Let $A = (\mathbf{R}[X, Y, Z])/(X^2 + Y^2 + Z^2 - 1)$ and (P, χ) be an oriented rank 2 projective module as above. We choose as above isomorphisms $\beta_1 : A_{z+1}^2 \xrightarrow{\sim} P_{z+1}$ and $\beta_2 : A_{z+1}^2 \xrightarrow{\sim} P_{z-1}$ such that $\wedge^2 \beta_1 = \chi$ and $\wedge^2 \beta_2 = \chi$. Let $\tau = \beta_2^{-1} \beta_1 \in SL_2(A_{(z-1)(z+1)})$. Let S^1 be the equator of the two sphere, that is,

$$S^1 = \{(a, b, 0) \mid \ a^2 + b^2 = 1\}.$$

Then, we have a continuous map $S^1 \to SL_2(\mathbf{R})$ induced by τ. We continue to denote this map by τ. Let $\phi : E(A) \to \mathbf{Z}$ be the homomorphism defined in Section 7. With the above notation we have

Theorem 8.4. $\phi(e(P,\chi)) = \text{degree}\,(\tau)$.

The proof of this theorem is similar to the proof of Theorem 6.7. There, we proved that $\phi(A^2,\chi) = 0 = \text{degree}\,(\tau)$, where τ is the continuous map $S^1 \to SL_2(\mathbf{R})$ which sends every element S^1 to the identity matrix.

The above theorem can be viewed as a generalization and its proof is similar. Therefore, we do not give all the details.

Proof of the Theorem. We choose a surjection $\alpha : P \to \cap_{i=1}^{s} m_i$ such that none of the points corresponding to m_i lie on the equator. (See the following lemma.) Also, we assume for simplicity that all the m_i are real. Let $e_1 = (1,0)$ and $e_2 = (0,1)$. Let $\cap_{i=1}^{s} m_i = J$. We have surjections

$$\alpha\beta_1 : A^2_{z+1} \twoheadrightarrow J_{z+1}$$

and

$$\alpha\beta_2 : A^2_{z-1} \twoheadrightarrow J_{z-1}.$$

Let

$$\begin{aligned} \alpha\beta_1(e_1) &= f_1, \qquad & \alpha\beta_1(e_2) &= f_2 \\ \alpha\beta_2(e_1) &= g_1, \qquad & \alpha\beta_2(e_2) &= g_2. \end{aligned}$$

Then, we have

$$e(P,\chi) = \sum_{i=1}^{s} (m_i, \omega_{m_i}),$$

where ω_{m_i} is the orientation of m_i given by $\bar{f}_1 \wedge \bar{f}_2$ or $\bar{g}_1 \wedge \bar{g}_2$ depending on whether the point corresponding to m_i is different from $(0,0,-1)$ or $(0,0,1)$.

Assume that the points corresponding to the maximal ideals $m_1, \dots,$ m_r lie on the upper hemisphere of the two sphere and the points corresponding to $m_{r+1}, \dots, m_s$ lie in the lower hemisphere. For $1 \le i \le r$, let

$$u_i = \det \begin{pmatrix} \dfrac{\partial f_1}{\partial X}(a_i,b_i,c_i), & \dfrac{\partial f_1}{\partial Y}(a_i,b_i,c_i), & \dfrac{\partial f_1}{\partial Z}(a_i,b_i,c_i) \\ \dfrac{\partial f_2}{\partial X}(a_i,b_i,c_i), & \dfrac{\partial f_2}{\partial Y}(a_i,b_i,c_i), & \dfrac{\partial f_2}{\partial Z}(a_i,b_i,c_i) \\ a_i, & b_i, & c_i \end{pmatrix},$$

where (a_i, b_i, c_i) is the point of S^2 corresponding to m_i. For $r+1 \leq i \leq s$, let

$$u_i = \det \begin{pmatrix} \frac{\partial g_1}{\partial X}(a_i, b_i, c_i), & \frac{\partial g_1}{\partial Y}(a_i, b_i, c_i), & \frac{\partial g_1}{\partial Z}(a_i, b_i, c_i) \\ \frac{\partial g_2}{\partial X}(a_i, b_i, c_i), & \frac{\partial g_2}{\partial Y}(a_i, b_i, c_i), & \frac{\partial g_2}{\partial Z}(a_i, b_i, c_i) \\ a_i, & b_i, & c_i \end{pmatrix},$$

where (a_i, b_i, c_i) is the point of S^2 corresponding to m_i. We see that $\phi(e(P,\chi)) = \sum_{i=1}^{s} \text{sign}\,(u_i)$.

Now, let $\sigma = \beta_1^{-1}\beta_2 \in SL_2(A_{(z-1)(z+1)})$. Then,

$$\sigma^T \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}.$$

We have a continuous map $S^1 \to SL_2(\mathrm{R})$ induced by σ^T. We continue to denote this map by σ^T. Now, using Lemma 8.2 and an argument similar to Theorem 6.7, we see that $\phi(e(P,\chi)) = \deg(\sigma^T)$. It is easy to see using the Gram-Schmidt process that degree$(\sigma^T) = -\deg(\sigma)$. Thus, $\phi(e(P,\chi)) = -\text{degree}\,(\sigma)$.

Let $\tau = \beta_2\beta_1^{-1}$ and $\tau : S^1 \to SL_2(\mathbf{R})$ be the induced continuous map. Since $\sigma = \beta_1\beta_2^{-1} = \tau^{-1}$, it follows easily using the Gram-Schmidt process that degree$\,(\tau) = -$degree$\,(\sigma)$. Hence, $\phi(e(P,\chi)) = $ degree$\,(\tau)$. This proves the theorem. □

We now prove a lemma that was used in the proof of the previous theorem.

Lemma 8.5. *Let $A = (\mathbf{R}[X,Y,Z])/(X^2+Y^2+Z^2-1)$ and P be a projective A-module of rank 2. Then, there exists a surjection $\alpha : P \twoheadrightarrow \cap_{i=1}^{s} m_i$ such that none of the points corresponding to the maximal ideals m_i, lie on the equator.*

Proof. Since the determinant of P is trivial, the determinant of the (A/zA)-module (P/zP) is trivial. Since $(A/zA) = (\mathbf{R}[X,Y])/(X^2 + Y^2 - 1)$ is a Dedekind domain, the module (P/zP) is free of rank 2. Therefore, we can choose a surjection

$$\overline{\beta} : \frac{P}{zP} \twoheadrightarrow \frac{A}{zA}.$$

We lift $\overline{\beta}$ to a linear map $\beta : P \to A$. We then have $\beta(P) + zA = A$.

By Swan's Bertini theorem (2.6), we can choose a linear map $\beta' : P \to A$ such that $\alpha = \beta + z\beta'$ satisfies the property that $\alpha(P) = \cap_{i=1}^{s} m_i$. Further, since $\beta(P) + zA = A$, we have $\alpha(P) + zA = A$. Hence, none of the points corresponding to the maximal ideals m_i lie on the equator. Hence the lemma is proved. □

Remark 8.6. Let A be the ring of polynomial functions on a smooth real orientable variety X of dimension n. Let P be a projective$A$-module of rank n and trivial determinant, and let χ be an orientation of P. Then (as in subsection 8.4) one can define an element $\varphi(e(P, \chi))$, whose image lands in $H^n(X(\mathbf{R}), \mathbf{Z})$. This element is simply the Euler class of the topological vector bundle associated to P.

9. Some examples.

Example 1. The Hopf bundle. Let $A = (\mathbf{R}[X, Y, Z])/(X^2 + Y^2 + Z^2 - 1)$, and let m be a maximal ideal of A corresponding to a real point. It is well known and follows from Theorem 4.1 that m is not generated by two elements.

We begin this section by giving an alternative simple proof in the spirit of this paper of this fact. The proof uses the following lemma which is of independent interest (see [**3**, Lemma 2.3]).

Lemma 9.1. *Let A be a ring. Let $J = (f, g) \subset A$ be an ideal. Suppose $J = (f', g')$. Suppose, further, that there exists a matrix*

$$\alpha = \begin{pmatrix} \lambda_{11}, & \lambda_{12} \\ \lambda_{21}, & \lambda_{22} \end{pmatrix}$$

in $M_2(A)$ such that $\det(\alpha) = 1$ *modulo J and*

$$\alpha \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} f' \\ g' \end{pmatrix}.$$

Then there exists a matrix $\beta \in SL_2(A)$ such that $\beta = \alpha$ modulo J and

$$\beta \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} f' \\ g' \end{pmatrix}.$$

Proof. Let $\det(\alpha) = 1 + h$, $h \in J$.

Let

$$\beta = \begin{pmatrix} \lambda_{11} + cg, & \lambda_{12} - cf \\ \lambda_{21} + dg, & \lambda_{22} - df \end{pmatrix}$$

where $c, d \in A$ will be chosen later. Then $\beta = \alpha$ modulo J and

$$\beta \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} f' \\ g' \end{pmatrix}.$$

Using the fact that

$$\alpha \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} f' \\ g' \end{pmatrix},$$

it follows by an easy computation that $\det(\beta) = 1 + h + cg' - df'$. Now, since $J = (f', g')$ and $h \in J$, we can choose $c, d \in A$ so that $h = df' - cg'$. With the above choice of c, d we have $\det(\beta) = 1$. This proves the lemma.

Theorem 9.2. *Let $A = (\mathbf{R}[X, Y, Z])/(X^2 + Y^2 + Z^2 - 1)$. Let m be a maximal ideal of A corresponding to a real point. Then m is not generated by 2 elements.*

Proof. Without loss of generality, we may assume that $m = (x, y, z - 1)$. Since $(z + 1)(z - 1) \in (x, y)$ and $z + 1 \notin m$, it follows that $(x, y) + m^2 = m$. Since $(z + 1)(z - 1) \in (x, y)$, it also follows that

$$m_{z+1} = (x, y).$$

Since $z - 1 \in m$, it follows that

$$m_{z-1} = (1, 0).$$

Since $x^2 + y^2 = 1 - z^2$, it follows that

$$\sigma = \begin{pmatrix} x, & -y/1 - z^2 \\ y, & x/1 - z^2 \end{pmatrix}$$

belongs to $SL_2(A_{(z-1)(z+1)})$ and

$$\sigma \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Now, assume to the contrary that m is generated by two elements f, g. We will derive a contradiction.

Since $m = (f, g)$, we have $(f, g) + m^2 = m$ and $\bar{f}, \bar{g}$ is a basis of the A/m-vector space m/m^2. Since $\overline{x}, \overline{y}$ also form a basis of m/m^2, it follows that there exists an element τ belonging to $GL_2(A/m) = GL_2(\mathbf{R})$ such that

$$(*) \qquad \tau \begin{pmatrix} \bar{f} \\ \bar{g} \end{pmatrix} = \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix}.$$

Let $\det(\tau) = \lambda \in \mathbf{R}^*$. Replacing the set of generators (f, g) of m by $(\lambda f, g)$, we may assume that $\tau \in SL_2(A/m)$ and satisfies $*$.

Since $m_{z+1} = (x, y) = (f, g)$, it follows now from Lemma 9.1 that there exists a τ_1 belonging to $SL_2(A_{z+1})$ such that

$$\tau_1 \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Now, since $z - 1 \in m$, $m_{z-1} = A_{z-1} = (f, g) = (1, 0)$. It follows that there exists an element τ_2 belonging to $SL_2(A_{z-1})$ such that

$$\tau_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}.$$

Therefore, we have

$$\tau_1 \tau_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Since

$$\sigma \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix},$$

we have

$$\tau_2^{-1} \tau_1^{-1} \sigma \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Therefore,

$$\tau_2^{-1} \tau_1^{-1} \sigma = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} = \tau_3,$$

where $\lambda \in A_{(z-1)(z+1)}$. We have $\sigma = \tau_1 \tau_2 \tau_3$.

Let $W = S^2 - \{(0,0,1),(0,0,-1)\}$. Since $\sigma \in SL_2(A_{(z-1)(z+1)})$, σ gives rise in a natural way to a map which we continue to denote by $\sigma : W \to SL_2(\mathbf{R})$. Let S^1 be the equator of S^2 that is

$$S^1 = \{(a,b,0) \mid a^2 + b^2 = 1\}.$$

Restricting σ to the equator we obtain $\sigma : S^1 \to SL_2(\mathbf{R})$ given by

$$\sigma(\cos\theta, \sin\theta, 0) = \begin{pmatrix} \cos\theta, & -\sin\theta \\ \sin\theta, & \cos\theta \end{pmatrix}.$$

Let $V = S^2 - \{(0,0,-1)\}$. Since $\tau_1 \in SL_2(A_{z+1})$, we obtain as above a continuous map

$$\tau_1 : V \longrightarrow SL_2(\mathbf{R}).$$

Since V is contractible, it follows that τ_1 is homotopic to the constant map.

Let $U = S^2 - \{(0,0,1)\}$. Since $\tau_2 \in SL_2(A_{z-1})$, we have a continuous map $\tau_2 : U \to SL_2(\mathbf{R})$ which is also homotopic to a constant. Therefore, the restrictions of τ_1 and τ_2 to S^1 are homotopic to constant maps.

We recall that

$$\tau_3 = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix},$$

where $\lambda \in A_{(z-1)(z+1)}$. We obtain a continuous map

$$\tau_3 : W \longrightarrow SL_2(\mathbf{R}),$$

where $W = U \cap V$.

The restriction of τ_3 to S^1, $\tau_3 : S^1 \to SL_2(\mathbf{R})$ is homotopic to a constant map, the homotopy given by the map

$$S^1 \times I \longrightarrow SL_2(\mathbf{R})$$

sending $((\cos\theta, \sin\theta, 0), t)$ to

$$\begin{pmatrix} 1, & \lambda(\cos\theta, \sin\theta, 0)t \\ 0, & 1 \end{pmatrix}.$$

Therefore, the continuous map

$$\tau_1\tau_2\tau_3 : S^1 \longrightarrow SL_2(\mathbf{R})$$

is homotopic to a constant map. Since $\tau_1\tau_2\tau_3 = \sigma$, we have the map

$$\sigma : S^1 \longrightarrow SL_2(\mathbf{R})$$

given by

$$\sigma(\cos\theta, \sin\theta, 0) = \begin{pmatrix} \cos\theta, & -\sin\theta \\ \sin\theta, & \cos\theta \end{pmatrix}$$

is homotopic to a constant. This is a contradiction since σ has degree 1. Therefore, m is not generated by two elements. □

Remark 9.3. Let $A = (\mathbf{R}[X, Y, Z])/(X^2 + Y^2 + Z^2 - 1)$, and let $m \subset A$ be a maximal ideal corresponding to a real point of S^2. Then there exists an A-linear surjection $P \twoheadrightarrow m$, where P is a projective A-module of rank 2. The projective A-module with the above property is determined uniquely into isomorphism and is called the Hopf bundle. Since m is not generated by 2 elements, the Hopf bundle is not trivial. If χ is an orientation of P (that is a generator of $\wedge^2 P$), then the value of $e(P, \chi) \in E(A) \xrightarrow{\sim} \mathbf{Z}$ is $+1$ or -1 (depending on the choice of orientation of P). This gives another proof of the fact that m is not generated by 2 elements. In fact, for any orientation ω_m of m, $\varphi(m, \omega_m) = \pm 1$, and if m is generated by 2 elements then $\varphi(m, \omega_m) = 0$.

Example 2. The tangent bundle of the two sphere. We consider the classical example of the tangent bundle of the two sphere and use the methods of this paper to show that it is nontrivial. We begin with some preliminaries.

Definition 9.4. Let A be a commutative ring with unity. A row $[a_1, \ldots, a_n] \in A^n$ is said to be unimodular (of length n) if there exist elements $b_1, \ldots, b_n \in A$ such that $a_1b_1 + a_2b_2 + \cdots + a_nb_n = 1$.

The set of unimodular rows of length n (with entries in A) is denoted by $Um_n(A)$.

The groups $GL_n(A)$, $SL_n(A)$ and $E_n(A)$ act on $Um_n(A)$. If a matrix $\sigma \in GL_n(A)$ transforms $[a_1, \ldots, a_n] \in Um_n(A)$ to $[b_1, \ldots, b_n]$, we say that $[a_1, \ldots, a_n]\underset{GL_N(A)}{\widetilde{\ }}[b_1, \ldots, b_n]$.

The relation $\underset{GL_n(A)}{\widetilde{\ }}$ is an equivalence relation on the set $Um_n(A)$.

Similarly, one can define the equivalence relations $\widetilde{}_{SL_n(A)}$ and $\widetilde{}_{E_n(A)}$ on $Um_n(A)$.

Example 9.5. Let A be as above. Let $[a_1, \dots, a_n] \in Um_n(A)$. Then

$$[a_1, \dots, a_n] \widetilde{}_{E_n(A)} [a_1 + \lambda_2 a_2, \dots + \lambda_n a_n, a_2, \dots, a_n].$$

Let A be as above and $[a_1, \dots, a_n] \in Um_n(A)$. Then the A-module $P = A^n/[a_1, \dots, a_n]$ is projective of rank $n-1$.

The following lemma is standard and easy to prove.

Lemma 9.6. *Let A be as above, and let $[a_1, \dots, a_n] \in A^n$ be unimodular. Then the following are equivalent*:

(i) $[a_1, \dots, a_n]$ *is the first row of a matrix belonging to $SL_n(A)$.*

(ii) $[a_1, \dots, a_n]$ *is the first row of a matrix belonging to $GL_n(A)$.*

(iii) $[a_1, \dots, a_n] \widetilde{}_{GL_n(A)} [1, 0, \dots, 0]$.

(iv) $[a_1, \dots, a_n] \widetilde{}_{SL_n(A)} [1, 0, \dots, 0]$.

(v) *The projective module $P = A^n/[a_1, \dots, a_n]$ is free of rank $n-1$.*

Theorem 9.7. *Let $A = (\mathbf{R}[X, Y, Z])/(X^2 + Y^2 + Z^2 - 1)$. Then $(x, y, z) \in Um_3(A)$, but there does not exist a matrix in $SL_3(A)$ having first row (x, y, z).*

Proof. Since $x^2 + y^2 + z^2 = 1$, $(x, y, z) \in Um_3(A)$. Since $(x, y, z) \in Um_3(A)$, and $z - 1$ is a unit of A_{z-1}, $(z(z-1), x, y) \in Um_3(A_{z-1})$.

Adding $x^2 + y^2$ to $z(z-1)$, we get

$$(z(z-1), x, y) \widetilde{}_{E_3(A_{z-1})} (1 - z, x, y).$$

Therefore,

$$(z, x, y) \widetilde{}_{GL_3(A_{z-1})} (z(z-1), x, y) \widetilde{}_{E_3(A_{z-1})} (1 - z, x, y) \widetilde{}_{E_3(A_{z-1})} (1, 0, 0).$$

The last equivalence follows since $1 - z$ is a unit of A_{z-1}. We thus obtain the following explicit completion of the row (z, x, y) viz.

$$\sigma = \begin{pmatrix} z, & \dfrac{x}{z-1}, & \dfrac{-y}{z-1} \\ x, & -1, & 0 \\ y, & 0, & 1 \end{pmatrix} \in SL_3(A_{z-1}).$$

Similarly, we obtain the matrix

$$\tau = \begin{pmatrix} z, & \dfrac{-x}{z+1}, & \dfrac{-y}{z+1} \\ x, & 1, & 0 \\ y, & 0, & 1 \end{pmatrix} \in SL_3(A_{z+1}).$$

Now, suppose to the contrary that $\beta \in SL_3(A)$ has first column (z, x, y). Since σ and τ both have (z, x, y) as the first column,

$$\sigma^{-1}\tau = \begin{pmatrix} 1 & * & * \\ 0 & \alpha & \\ 0 & & \end{pmatrix},$$

where $\alpha \in SL_2(A_{z-1)(z+1)})$. Now, $\sigma^{-1}\tau = \sigma^{-1}\beta \cdot \beta^{-1}\tau$. Now, since the first column of β is (z, x, y),

$$\sigma^{-1}\beta = \begin{pmatrix} 1 & * & * \\ 0 & \alpha_1 & \\ 0 & & \end{pmatrix},$$

where $\alpha_1 \in SL_2(A_{z-1})$ and

$$\beta^{-1}\tau = \begin{pmatrix} 1 & * & * \\ 0 & \alpha_2 & \\ 0 & & \end{pmatrix},$$

where $\alpha_2 \in SL_2(A_{z+1})$. Since $\sigma^{-1}\tau = \sigma^{-1}\beta \cdot \beta^{-1}\tau$, we have $\alpha = \alpha_1\alpha_2$. Let $U = S^2 - \{(1, 0, 0)\}$, $V = S^2 - \{(0, 0, 1)\}$.

Since $\alpha \in SL_3(A_{(z-1)(z+1)})$, $\alpha_1 \in SL_3(A_{z-1})$ and $\alpha_2 \in SL_3(A_{z+1})$, we have continuous maps

$$\begin{aligned} \alpha : U \cap V &\longrightarrow SL_2(\mathbf{R}) \\ \alpha_1 : U &\longrightarrow SL_2(\mathbf{R}) \\ \alpha_2 : V &\longrightarrow SL_2(\mathbf{R}). \end{aligned}$$

Since U and V are contractible, the restrictions of α_1 and α_2 to the equator of the two sphere are homotopic to constant maps. Since $\alpha = \alpha_1 \cdot \alpha_2$, we deduce that the restriction of α to the equator is homotopic to a constant map. We will show that this yields a contradiction. Since

$$\sigma^{-1}\tau = \begin{pmatrix} 1, & *, & * \\ 0, & \alpha & \\ 0, & & \end{pmatrix},$$

an explicit computation shows that the restriction of α to the equator is given by the map $\alpha : S^1 \to SL_2(\mathbf{R})$ defined as follows:

$$\alpha(\cos\theta, \sin\theta, 0) = \begin{pmatrix} \cos(2\theta), & -\sin(2\theta) \\ \sin(2\theta), & \cos(2\theta) \end{pmatrix}.$$

The map α has degree -2 and hence is not homotopic to a constant. This is a contradiction and proves that the unimodular row $(x, y, z) \in A^3$ is not completable to a matrix belonging to $SL_3(A)$. □

Corollary 9.8. *Let $A = (\mathbf{R}[X, Y, Z])/(X^2 + Y^2 + Z^2 - 1)$. Let $P = A^3/(x, y, z)$. Then P is projective A-module of rank 2 but not free.*

Remark 9.9. We can give another proof of Corollary 9.8 in the following manner. Let $P = A^3/(x, y, z)$. Let e_i denote the images in P of the standard basis elements in A^3. Then $\chi = xe_2 \wedge e_3 + ye_3 \wedge e_1 + ze_1 \wedge e_2$ is a generator of $\wedge^2 P$. Let $s : P \to A$ be the A-linear map defined by $s(e_1) = 0, s(e_2) = Z, s(e_3) = -y$. Then we compute the Euler class of (P, χ) using s to be the image of the element $x(y \wedge z)$ of $E(A)$ A computation shows that $\varphi(e(P, \chi))$ is the integer 2. Therefore, P cannot be free, otherwise $\varphi(e(P, \chi)) = 0$.

10. On nonreduced oriented zero cycles. Let $A = (\mathbf{R}[X, Y, Z])/(h(X, Y, Z))$ be a regular affine domain of dimension 2 over $\mathbf{R}$. Let $X = \operatorname{Spec} A$, and suppose that $X(\mathbf{R})$ has t compact connected components $M_1, \ldots, M_t$. We recall that in Section 7 we have defined a homomorphism $\phi : E(A) \to \mathbf{Z}^t$. We recall how ϕ is defined. If m is a complex maximal ideal of A, we set $\phi((m, w_m)) = 0$. If m is a real maximal ideal of A such that the associated real point belongs to a noncompact connected component of $X(\mathbf{R})$, we set $\phi(m, w_m) = 0$.

Let m be a real maximal ideal of A corresponding to a point (a,b,c) belonging to a compact connected component M_i of $X(\mathbf{R})$, and let w_m be a generator of $\wedge^2 m/m^2$ given by $\bar{f}\wedge\bar{g}$, where $\bar{f},\bar{g}$ generate m/m^2. Let

$$u = \det\begin{pmatrix} \dfrac{\partial f}{\partial X}(a,b,c), & \dfrac{\partial f}{\partial Y}(a,b,c), & \dfrac{\partial f}{\partial Z}(a,b,c) \\ \dfrac{\partial g}{\partial X}(a,b,c), & \dfrac{\partial g}{\partial Y}(a,b,c), & \dfrac{\partial g}{\partial Z}(a,b,c) \\ \dfrac{\partial h}{\partial X}(a,b,c), & \dfrac{\partial h}{\partial Y}(a,b,c), & \dfrac{\partial h}{\partial Z}(a,b,c) \end{pmatrix}.$$

We set $\phi((m,w_m)) = (0,0,\dots,0,1,0,\dots,0)$, (where 1 is in the ith place) if u is positive. We set $\phi((m,w_m)) = (0,0,\dots,0,-1,0,\dots,0)$, (where -1 is in the ith place) if u is negative. Thus, we obtain a homomorphism $\phi : E(A)\to\mathbf{Z}^t$.

More generally, let $J\subset A$ be an ideal of height 2 which is not necessarily reduced such that J/J^2 is generated by two elements $\bar{f},\bar{g}$, and let w_J be the "orientation of J" (that is, the generator of $\wedge^2 J/J^2$) given by $\bar{f}\wedge\bar{g}$. To the pair (J,w_J), we assign an element of $\mathbf{Z}^t$ as follows:

By Theorem 2.7 we can choose lifts f,g of $\bar{f},\bar{g}$ which satisfy the property that $(f,g)=J\cap J'$, where either $J'=A$ or $J'=m_1\cap\cdots\cap m_r$, where m_i are distinct maximal ideals of A satisfying $m_i+J=A$, $1\le i\le r$. If $J'=A$, set $\phi(J,w_J)=0$; otherwise, we define $\phi((J,w_J)) = -\sum\phi((m_i,w_{m_i}))$, where w_{m_i} is the orientation of m_i given by $\bar{f}\wedge\bar{g}$. The function ϕ is well defined. (See [**1**, Remark 4.16] for details.) We have

Theorem 10.1. *Let $A=(\mathbf{R}[X,Y,Z])/(h(X,Y,Z))$ be a regular affine domain over $\mathbf{R}$ of dimension 2. Let $X=\operatorname{Spec}A$, and suppose that $X(\mathbf{R})$ has t compact connected components say $M_1,\dots,M_t$. Let $m\subset A$ be a maximal ideal such that the associated real point belongs to the compact connected component M_1. Let $J\subset A$ be an m primary ideal such that J/J^2 is generated by 2 elements, and let w_J be a generator of $\wedge^2 J/J^2$. Then $\phi(J,w_J)=(b,0,\dots,0)$ for some integer $b\in\mathbf{Z}$.*

Proof. We only give a sketch, as the proof is similar to that of Theorem 5.13. Let $w_J=\bar{f}\wedge\bar{g}$, where $\bar{f},\bar{g}$ generate J/J^2. We choose by Theorem 2.7 lifts f,g of $\bar{f},\bar{g}$ which satisfy the property that $(f,g)=J\cap m_1\cap\cdots\cap m_r$, where $m_i+J=A$, $1\le i\le r$.

Replacing f by $f+\lambda g$, we may assume that by Swan's Bertini theorem 2.4, that $\dim A/f = 1$ and that $(A/f)_{\widetilde{m}}$ is regular for every maximal ideal $\widetilde{m}$ of A containing f different from m.

Let $A' = (\mathbf{R}[X,Y,Z])/(f(X,Y,Z), h(X,Y,Z))$ and $X' = \operatorname{Spec} A'$. By assumption, $A'_{\widetilde{m}}$ is regular for every maximal ideal $\widetilde{m}$ of A' different from m. Now, $X'(\mathbf{R})$ is a union of connected components. Some of these are contained in M_1. The rest are contained in $M_2 \cup \cdots \cup M_t$. Suppose M is a connected component of $X'(\mathbf{R})$ which is not contained in M_1. Since the real point corresponding to m belongs to M_1, and A' is locally regular at every maximal ideal different from m, it follows by the methods of Section 5 that M is diffeomorphic to S^1.

Now, suppose that amongst the maximal ideals $m_1, \ldots, m_r$, the points corresponding to $m_1, \ldots, m_q$ belong to M. Let m_i correspond to the point (a_i, b_i, c_i), and let

$$u_i = \det \begin{pmatrix} \frac{\partial f}{\partial X}(a_i,b_i,c_i), & \frac{\partial f}{\partial Y}(a_i,b_i,c_i), & \frac{\partial f}{\partial Z}(a_i,b_i,c_i) \\ \frac{\partial g}{\partial X}(a_i,b_i,c_i), & \frac{\partial g}{\partial Y}(a_i,b_i,c_i), & \frac{\partial g}{\partial Z}(a_i,b_i,c_i) \\ \frac{\partial h}{\partial X}(a_i,b_i,c_i), & \frac{\partial h}{\partial Y}(a_i,b_i,c_i), & \frac{\partial h}{\partial Z}(a_i,b_i,c_i) \end{pmatrix}.$$

Now, by imitating the proof of Theorem 5.13, it follows that

$$\sum_{i=1}^{q} \operatorname{sign}(u_i) = 0.$$

Arguing as above, it follows easily that $\phi(J, w_J)) = (b, 0, \ldots, 0)$ for some integer b. $\square$

The proof of the following theorem is quite similar to that of Theorem 10.1. We omit the proof.

Theorem 10.2. *Let $A = (\mathbf{R}[X,Y,Z])/(h(X,Y,Z))$ be a regular affine domain of dimension 2. Let $X = \operatorname{Spec} A$, and suppose that $X(r)$ has t compact connected components. Let $m \subset A$ be a maximal ideal such that the associated real point belongs to a noncompact connected component of $X(\mathbf{R})$. Let $J \subset A$ be an m-primary ideal such that J/J^2*

is generated by two elements, and let w_J be a generator of $\wedge^2 J/J^2$. Then we have $\phi((J, w_J)) = (0, 0, \ldots, 0, 0) \in \mathbf{Z}^t$.

Remark 10.3. More generally, we can prove analogous results in the case where $A = (\mathbf{R}[X_1, \ldots, X_{n+1}])/(h(X_1, \ldots, X_{n+1}))$ is an n-dimensional regular affine domain over $\mathbf{R}$.

Remark 10.4. The results of this section were obtained previously by Bhatwadekar and the author (unpublished). We proved versions of these results where A is an n-dimensional regular affine domain over $\mathbf{R}$ such that $K_A \approx A$. The proofs given here are a little different.

11. A question. Let A be a regular affine domain over an infinite perfect field k with $\dim A = 2$. Let $f, g \in A$ be nonzero elements such that the ideal $(f, g) = A$. Let $\sigma \in SL_2(A_{fg})$. Then, as in the clutching construction of vector bundles we can associate to σ, a rank 2 projective A-module by patching the free A_f-module A_f^2 and the free A_g-module A_g^2 via the cocycle σ. Further, since $\sigma \in SL_2(A_{fg})$, the generators $e_1 \wedge e_2$ of $\wedge^2 A_f^2$ and $\wedge^2 A_g^2$ patch to yield a generator χ of $\wedge^2 P$.

With the above notation we have the following question which is motivated by results in Section 8.

Question. Is the map $\phi : SL_2(A_{fg}) \to E(A)$ sending $\sigma \in SL_2(A_{fg})$ to $e(P, \chi)$ (the Euler class of the oriented projective module P) a homomorphism?

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