

Lecturer	Lectures	Hours	Emphasis
Cordian Riener	4	6	Opening lecture (Sturm, Hermite, guiding questions); symmetric semi-algebraic sets; sums of squares, spectrahedra, and symmetry reduction; hyperbolic polynomials
Manoj Keshari	3	4.5	Ordered and real closed fields; real spectrum, Artin–Lang, real Nullstellensatz; Hilbert’s 17th problem and Artin’s theorem
Saugata Basu	3	4.5	Quantifier elimination and Tarski–Seidenberg; cell decomposition and triangulation; Thom–Milnor bounds and quantitative Betti number bounds
Marie-Françoise Roy	3	4.5	Positivstellensätze; effective and quantitative bounds; algorithms in real algebraic geometry
Apoorva Khare	3	4.5	Putinar’s Positivstellensatz and the moment–SOS hierarchy; matrix positivity and preservers; total positivity, real-rootedness, and stability
Total	16	24	

Detailed lecture plan

Opening lecture (CR 0)

Lecture CR 0. A first tour of real algebraic geometry.

- Real versus complex: why real solution sets behave differently; the empty circle; nonnegative polynomials that are not sums of squares (Motzkin’s example).
- Counting real roots of a univariate polynomial: Descartes’ rule of signs, Sturm sequences, Hermite’s quadratic form and root counting by signature.
- First encounters with the main characters of the week: semi-algebraic sets, sums of squares, symmetric functions and power sums.
- A map of the week: the two guiding questions (describing real solution sets; certifying positivity) and how the three strands address them.

Real algebra: ordered fields, real closed fields, and the real spectrum (MK 1–3)

Lecture MK 1. Ordered fields and real closed fields.

- Orderings, positive cones, and preorderings of fields; formally real fields; sums of squares in fields.
- Artin–Schreier theory: existence and extension of orderings.
- Real closed fields: definition, equivalent characterizations, real closure of an ordered field.
- Examples: $\mathbb{R}$, real algebraic numbers, Puiseux series fields, non-Archimedean real closed fields.

Lecture MK 2. The real spectrum, the real Nullstellensatz, and the Artin–Lang theorem.

- The real spectrum of a commutative ring: points, topology, specializations; comparison with the Zariski spectrum.

- Real radical ideals and the real Nullstellensatz.
- The Artin–Lang homomorphism theorem and the Tarski principle from the algebraic side.
- Examples: real spectra of polynomial rings in one and two variables.

Lecture MK 3. Hilbert’s 17th problem and Artin’s theorem.

- Nonnegative polynomials versus sums of squares of polynomials; Hilbert’s 1888 theorem and the classical counterexamples.
- Artin’s theorem: nonnegative polynomials are sums of squares of rational functions; proof via Artin–Lang.
- Positivity on semi-algebraic sets: first examples of certificates, preparing the Positivstellensätze of MFR 1.

Semi-algebraic geometry and quantitative topology (SB 1–3)

Lecture SB 1. Semi-algebraic sets, quantifier elimination, and the Tarski–Seidenberg theorem.

- Semi-algebraic sets and semi-algebraic maps; basic constructions.
- The first-order language of ordered fields; quantifier elimination; the Tarski transfer principle.
- Projection of semi-algebraic sets and the Tarski–Seidenberg theorem.
- Consequences: closure, interior, images under polynomial maps, and dimension are semi-algebraic.

Lecture SB 2. Cell decomposition, triangulation, and local structure.

- Cylindrical decomposition and the cell decomposition theorem.
- Triangulation; semi-algebraic connected components; the curve selection lemma; local conic structure.
- Finiteness of topological invariants; examples illustrating the transition from algebraic equations to semi-algebraic geometry.

Lecture SB 3. Thom–Milnor bounds and quantitative Betti number bounds.

- Betti numbers as measures of topological complexity; the Morse-theoretic ideas behind degree-dependent bounds.
- Thom–Milnor bounds for real algebraic hypersurfaces and varieties.
- Single-exponential bounds for semi-algebraic sets defined by s polynomials of degree at most d in k variables; bounds for projections, arrangements, and sign conditions.
- Semi-algebraic sets with group actions, equivariant triangulations, equivariant homology, and quantitative bounds in the presence of symmetry.

Positivstellensätze: certificates, effectivity, and algorithms (MFR 1–3)

Lecture MFR 1. Positivstellensätze.

- Preorderings and quadratic modules in polynomial rings.

- The Krivine–Stengle Positivstellensatz; the real Nullstellensatz as a special case; certificates for emptiness, infeasibility, and positivity over basic closed semi-algebraic sets.
- Schmüdgen’s Positivstellensatz and the Archimedean condition; limitations and geometric obstructions.

Lecture MFR 2. Effective and quantitative Positivstellensätze.

- Degree bounds and size bounds in Positivstellensatz certificates.
- Dependence on the number of variables, number of constraints, degrees, coefficient size, and geometric format.
- Effective real Nullstellensatz and Positivstellensatz bounds in algorithmic applications; comparison with semidefinite relaxations.

Lecture MFR 3. Algorithms in real algebraic geometry.

- Sign conditions, Thom encodings, and real root isolation over real closed fields.
- Cylindrical algebraic decomposition and algorithmic quantifier elimination; critical point methods.
- Complexity: doubly versus singly exponential algorithms; connections between algorithms and certificates.

Positivity in polynomials and matrices (AK 1–3)

Lecture AK 1. Putinar’s Positivstellensatz and the moment–SOS hierarchy.

- Archimedean quadratic modules and Putinar’s Positivstellensatz; comparison with Schmüdgen’s theorem.
- The moment problem viewpoint; the moment–SOS hierarchy in polynomial optimization and its convergence.
- Examples bridging classical inequalities and modern certificates.

Lecture AK 2. Matrix positivity and preservers.

- Positive semidefinite matrices; the Schur product theorem.
- Entrywise and functional preservers of positivity: the theorems of Schoenberg and Rudin; preservers in fixed dimension.
- Links between matrix positivity and polynomial positivity.

Lecture AK 3. Total positivity, real-rootedness, and stability.

- Totally positive matrices and determinantal inequalities; Pólya frequency sequences.
- Real-rooted polynomials, stability, and positivity-preserving transformations.
- Symmetric functions and positivity; connections with hyperbolic polynomials (CR 3) and a synthesis of the positivity themes of the week.

Symmetry, convexity, and hyperbolicity (CR 1–3)

Lecture CR 1. Symmetric algebraic and semi-algebraic sets.

- Actions of finite groups, especially symmetric groups, on polynomial rings and real varieties; invariants and orbit spaces.
- Degree principles and half-degree principles for symmetric polynomials.
- Topology of symmetric real algebraic and semi-algebraic sets and its equivariant description (continuing SB 3).

Lecture CR 2. Sums of squares, spectrahedra, and symmetry reduction.

- Sums-of-squares cones, moment cones, spectrahedra, spectrahedral shadows, and linear matrix inequalities.
- Semidefinite programming relaxations for polynomial optimization.
- Exploiting group actions to block-diagonalize and reduce semidefinite programs; examples from symmetric polynomial optimization.

Lecture CR 3. Hyperbolic polynomials and hyperbolicity cones.

- Hyperbolic homogeneous polynomials and hyperbolicity with respect to a direction; hyperbolicity cones and Gårding's convexity theorem.
- Examples: the determinant, the elementary symmetric polynomials, and discriminantal families.
- Determinantal representations, spectrahedra, and the generalized Lax conjecture; open problems and directions for further exploration.

Topic coverage checklist

Topic	Where it appears in the plan
Basic theory of real closed fields	Lecture MK 1
Real spectrum, real Nullstellensatz, Artin–Lang	Lecture MK 2
Hilbert’s 17th problem and Artin’s theorem	Lecture MK 3
Quantifier elimination and Tarski–Seidenberg theorem	Lecture SB 1
Geometry of semi-algebraic sets	Lectures SB 1–2, CR 1
Cell decomposition theorem	Lecture SB 2
Quantitative bounds on Betti numbers; Thom–Milnor and extensions	Lecture SB 3
Positivstellensatz	Lecture MFR 1
Effective/quantitative Positivstellensatz	Lecture MFR 2
Algorithms in real algebraic geometry	Lectures CR 0, MFR 3
Putinar’s Positivstellensatz	Lecture AK 1
Positivity in matrices	Lectures AK 2–3
Sums of squares and spectrahedra	Lectures MK 3, MFR 1, CR 2
Topology of symmetric algebraic sets	Lecture CR 1
Equivariant homology of semi-algebraic sets	Lectures SB 3 and CR 1
Hyperbolicity of polynomials	Lectures CR 3, AK 3
Random and metric algebraic geometry	Not covered in this workshop; proposed for the advanced follow-up workshop

Exercise and discussion sessions

The daily exercise sessions (16.15–17.15) are run by the tutors in coordination with the speakers, in the standard NCM Workshop format. Each session comes with a problem sheet prepared in advance and treats the lectures of the same day. Depending on the local schedule, the session on Thursday may be replaced by an excursion.

Session	Content
Tutorial 1 (Mon; CR 0, MK 1–2)	Sturm sequences and Hermite forms computed by hand; orderings of fields; examples of real closed fields and real closures.
Tutorial 2 (Tue; MK 3, SB 1–2)	Real spectra of simple rings; sums of squares of rational functions; quantifier elimination and projections in small examples; semi-algebraic descriptions of natural sets.
Tutorial 3 (Wed; SB 3, MFR 1–2)	Cell decompositions and Betti number computations; first Positivstellensatz certificates; examples of degree bounds.
Tutorial 4 (Thu; MFR 3, AK 1, CR 1)	Algorithms on small inputs (sign conditions, cylindrical decomposition); Putinar certificates and the moment–SOS hierarchy in low degree; symmetric and equivariant computations.
Tutorial 5 (Fri; CR 2–3, AK 2)	Spectrahedra and symmetry-reduced semidefinite programs; hyperbolicity cones of elementary symmetric polynomials; entrywise preservers; poster session for participants.
Open problems (Sat; after AK 3)	Short open-problem presentations by lecturers, tutors, and participants, including problems on total positivity and real-rootedness arising from the final lecture; concluding discussion.

Expected participant outcomes

By the end of the workshop, participants should be able to:

- count and locate real roots of univariate polynomials using Sturm sequences and Hermite forms;
- explain the role of ordered and real closed fields, the real spectrum, and quantifier elimination in real algebraic geometry;
- use the Tarski–Seidenberg theorem to prove that natural constructions preserve semi-algebraicity;
- understand the difference between nonnegativity and sums-of-squares representations, and prove Artin’s theorem;
- recognize and construct Positivstellensatz-type certificates, including effective and quantitative versions and Putinar’s theorem;
- state and interpret Thom–Milnor type bounds and modern quantitative bounds for Betti numbers;
- describe how symmetry changes both the topology and the computation of semi-algebraic sets;
- identify basic examples of spectrahedra and hyperbolicity cones, and formulate research-level questions at the interface of positivity, convexity, and symmetry.

Organizational notes

- Each lecturer will provide a one-page abstract and a short list of prerequisites for their lectures, in the format required for the NCM web page (speakers and syllabus).
- A common set of notes will be assembled from the lecture outlines, with the exercise sheets contributed by the speakers and tutors, and made available to participants after the workshop.
- The time-table above is tentative; the order of lectures within a day and the placement of the excursion can be adjusted to the local schedule at IIT Bombay.
- Research-level talks on current topics in real algebraic geometry, including those not covered in the workshop, are planned for the LSC–NCM research meeting at IIT Bombay in the week following the workshop; workshop participants are welcome to attend.
- **Proposed follow-up.** In line with the committee’s suggestion, the organizers propose a second, advanced workshop in 2027 building on this one, with topics such as random and metric real algebraic geometry, convex algebraic geometry, o-minimality, moment problems, and computational implementations. Participants of the present workshop would form its natural audience.

Suggested background references

The following references are intended as a starting point for participants. Lecture-specific references will be added by the individual speakers.

References

- [1] J. Bochnak, M. Coste, M.-F. Roy, *Real Algebraic Geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete 36, Springer, 1998.
- [2] S. Basu, R. Pollack, M.-F. Roy, *Algorithms in Real Algebraic Geometry*, Algorithms and Computation in Mathematics 10, 2nd ed., Springer, 2006.
- [3] T. Y. Lam, An introduction to real algebra, *Rocky Mountain J. Math.* 14 (1984), 767–814.
- [4] M. Marshall, *Positive Polynomials and Sums of Squares*, Mathematical Surveys and Monographs 146, American Mathematical Society, 2008.
- [5] A. Prestel, C. Delzell, *Positive Polynomials: From Hilbert’s 17th Problem to Real Algebra*, Springer Monographs in Mathematics, Springer, 2001.
- [6] C. Scheiderer, *A Course in Real Algebraic Geometry: Positivity and Sums of Squares*, Graduate Texts in Mathematics 303, Springer, 2024.
- [7] J.-B. Lasserre, *Moments, Positive Polynomials and Their Applications*, Imperial College Press, 2010.
- [8] G. Blekherman, P. Parrilo, R. Thomas (eds.), *Semidefinite Optimization and Convex Algebraic Geometry*, MOS-SIAM Series on Optimization 13, SIAM, 2013.
- [9] A. Khare, *Matrix Analysis and Entrywise Positivity Preservers*, London Mathematical Society Lecture Note Series 471, Cambridge University Press, 2022.