Orbital measures and spline functions

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Consider a Hermitian $n \times n$-matrix X with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$, and the projection Y of X on the $(n-1)\times(n-1)$ upper left corner. Rayleigh Theorem says that the eigenvalues $\mu_1 \leq \cdots \leq \mu_{n-1}$ of Y interlace those of X:

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \cdots \leq \mu_{n-1} \leq \lambda_n.$$

If the matrix X is distributed uniformly on a $U(n)$-orbit, then the joint distribution of the eigenvalues $\mu_1, \ldots, \mu_{n-1}$ is described by a formula due to Baryshnikov. More generally the eigenvalues of the projection of X on the $k \times k$ upper left corner $(1 \leq k \leq n-1)$ is distributed according to a determinantal formula due to Olshanski. In particular (for $k = 1$) the entry X_{11} is distributed according to a probability measure on R, the density of which is a spline function, as observed by Okounkov. In other words these results describe, for the action of the unitary group $U(n)$ on the space $\mathcal{H}_n$ of $n \times n$ Hermitian matrices, the projection of orbital measures on the subspaces $\mathcal{H}_k$.

Hence this topic involves harmonic analysis for compact Lie groups, and the analysis of spline functions whcih belongs to interpolation and approximation theory.

Recently analogous results have been obtained by Zubov in case of the action of the orthogonal group on the space of real skew-symmetric matrices. For the action of the orthogonal group on the space of real symmetric matrices much less is known.

A related topic is about the projection of an orbit, for the action of the unitary group $U(n)$ on the space $\mathcal{H}_n$ of $n \times n$ Hermitian matrices, on the subspace of diagonal matrices. Horn's theorem describe the image of this projection. the projection of the orbital measure has been studied by Heckman and Duistermaat.