A study of Suslin matrices: their properties and uses

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Abstract: We describe recent developments in the study of unimodular rows over a commutative ring by studying the associated group $\mathrm{SUm}_r(R)$, generated by Suslin matrices associated to a pair of rows v, w with $\langle v, w\rangle = 1$.

We also sketch some futuristic developments which we expect on how this association will help to solve a long standing conjecture of Bass–Suslin (initially in the metastable range, and later the entire expectation) regarding the completion of unimodular polynomial rows over a local ring, as well as how this study will lead to understanding the geometry and physics of the orbit space of unimodular rows under the action of the elementary subgroup.

1 Introduction

We begin by recapitulating the birth and early use of the Suslin matrices. The genesis is in the beautiful §5 of Suslin's paper [58]. He has said so much, with such fluency and consummate ease; it begets an area of mathematics rich in its connections with the rest of mathematics. The title of §5 'A procedure for constructing invertible matrices' is most intriguing. This section is also astounding in another sense; it is the first instance we know where Suslin has penned a flow of thoughts without much elaboration; as was his normal style. Naturally, it behoves his admirers to unearth the encrypted wisdom stored in it.

We intersperse this history with our own rambling thoughts of some of our immediate expectations. (A computer-algebra aided study, (especially wise with (perhaps) use of sparse matrices), will be helpful to ease some of our mendications.) We are prejudiced in choosing outlets which we feel will lead to a solution of two of the central problems in classical algebraic K-theory; both are questions regarding finding a procedure to complete a unimodular row to an invertible matrix, one of length d over a d dimensional affine algebra over an algebraically closed field (posed by Suslin), and the other of a unimodular polynomial row of any length over a local ring (posed by Bass–Suslin). We have made some progress in these directions, using the truncated Suslin matrices, and we refer the reader to [15] for the first problem, and [48], [49] for the second one. But the reader will feel the stirrings that the subject of the study of unimodular rows will soon evolve far beyond the range of these important classical problems.

We proceed to detail the association of a composition of two reflections $\tau_{(v,w)} \circ \tau_{(e_1,e_1)}$ with a pair of rows v, w with $\langle v, w\rangle = 1$. This association enables one to study the orbit space of unimodular rows under elementary action. Moreover since $\tau_{(v,w)} \circ \tau_{(e_1,e_1)}$ is an orthogonal transformation, one gets a homomorphism from $\mathrm{SUm}_r(R)$, the subgroup of the linear group generated by the Suslin matrices, to the special orthogonal group $\mathrm{SO}_{2(r+1)}(R)$; which is a well studied object. This allows us to pull back useful information in the study of unimodular rows.

The group $\mathrm{SUm}_r(R)$ has properties resembling those of classical spinor groups; and we feel that the further study of this group will lead to a better understanding of the geometry and physics of the orbit space of unimodular rows under the action of the elementary subgroup.

2 The Suslin Matrices

Given two rows v, $w \in \mathrm{M}_{1,r+1}(R)$, $r \geq 1$, in ([58], §5) Suslin associates with them a matrix $S_r(v, w) \in \mathrm{M}_{2^r}(R)$ of determinant $\langle v, w\rangle^{2^{r-1}} = (v \cdot w^t),^{2^{r-1}}$ whose entries are from the coordinates of v, w upto a sign. We call these the Suslin matrix w.r.t. v, w. They are particularly interesting to us when they are in $\mathrm{SL}_{2^r}(R)$, i.e. when $\langle v, w\rangle = v \cdot w^T t$ is 1. The explicit construction of the Suslin matrix is defered for the moment.

Trimurthi of Suslin Matrices

So far the Suslin matrix has manifested in at least three different contexts:

• Establishing that the unimodular row $(a_0, a_1, a_2^2, \ldots, a_r^r)$ can be completed to an invertible matrix. See the seminal paper of Suslin [58]; especially Theorem 2, Proposition 1.6 and the beautiful §5.

• From studying the Koszul complex associated to a unimodular row. See [65], Section 2, especially Proposition 2.2, Corollary 2.5.

• As orthogonal transformations on a certain space. See ([25], Corollary 4.2).

More recent developments

Two recent developments are briefly mentioned here. The reader should refer to the cited texts for notations which have not been explained here.

The Fundamental property of Suslin matrices in [24] led the referee to suspect a link between Suslin matrices and Spin groups. This connection was established in the thesis of Vineeth Chintala and appears in [12]. We sketch some of his ideas next.

For a commutative ring R, the hyperbolic space $H(R^n)$ is the module $R^n \times R^n$ endowed with a quadratic form q such that $q(v, w) == \langle v, w\rangle = v \cdot w^t$. To this structure one can associate the Clifford algebra $Cl_n(R)$ of the quadratic form, which is isomorphic to the matrix ring $M_{2^n}(R)$. Vineeth Chintala proved in [12] that the map $\varphi : H(R^n) \mapsto M_{2^n}(R)$ given by

$$\varphi(v, w) \quad = \quad \begin{pmatrix} 0 & S_{n-1}(v, w) \\ S_{n-1}(w, v)^t & 0 \end{pmatrix}$$

induces a R-algebra isomorphism. One can then derive the Jose–Rao fundamental property of Suslin matrices from this.

The map $S_{n-1}(v, w) \mapsto S_{n-1}(w, v)^t$ can be used to construct an involution $x \mapsto x^*$ on $Cl_n(R) = Cl_0(R) \oplus Cl_2(R)$. One defines the Spin groups

$$Spin_{2n}(R) \quad = \quad \{x \in Cl_0(R) \mid xx^* = 1 \text{ and } xH(R^n)x^{-1} = H(R^n)\}.$$

This involution on $Cl_n(R)$ corresponds to the standard involution on $M_{2^n}(R)$. One can define the groups

$$G_{n-1}(R) \quad = \quad \{g \in GL_{2^{n-1}} \mid gSg^* \text{is a Suslin matrix, for all Suslin matrices S}\}.$$

The subgroup of $G_{n-1}(R)$ consisting of those which preserve the quadratic form on $H(R^n)$ is denoted by $SG_{n-1}(R)$. Vineeth Chintala proves that there is an isomorphism $Spin_{2n}(R) \simeq SG_{n-1}(R)$.

The subgroup generated by the Suslin matrices is thus the rational points of a certain Spinor group.

The second new approach to Suslin matrices occurs in the work of Aravind Asok and Jean Fasel in []. Here there is an edge map interpretation for any regular algebra (with which 2 is invertible) in terms of Suslin matrices. We shall say a bit more about this later; but refer to [] for more details of this approach.

Use of Suslin Matrices

The Suslin matrices have proved useful in several contexts. The main application of Suslin matrices, so far, have been in the following directions:

• A unimodular row of the form $(a_0, a_1, a_2^2, \ldots, a_r^r)$ can be completed to a matrix $\beta_r(v, w)$, with $v = (a_0, a_1, a_2, \ldots, a_r)$, and w any row with $\langle v, w\rangle = 1$, of determinant one. (We may also just write this as $\beta_r(v)$ for brevity.)

Suslin mentions in ([58], §5) that a completion can be got by doing a series of row and column operations on the matrix $S_r(v, w)$ to reduce it to size $(r+1)$. However, an explicit process (as suggested by Suslin, based on the sparseness of the Suslin matrix) is **far from clear**, even in small sizes. A different reasoning justifes this in ([65], §2). Undoubtedly, ([58], Proposition 1.6) also gives a neat way of writing a completion, and also ties up with the Suslin matrix.

It would be both nice and useful if a good algorithm can be developed to get a $\beta_r(v,w)$ from a $S_r(v,w)$. We believe that an appropriate $\beta_r(v,w)$ will replicate the role played by $S_r(v,w)$. The actual use of a "nice" (and explicit) $\beta_2(v,w)$ can be seen in the works ([44], Lemma 2, Lemma 3), ([57], §5).

Note that it is unclear, and probably unjustified, to expect that any two $\beta_r(v,w)$ got from a $S_r(v,w)$ are equivalent in $\mathrm{E}_{r+1}(R)$. Indeed, there seem to be completions β of (a^2,b,c) which may not arise from a $S_2(v,w)$: The first completion of a unimodular row of the form (a^2,b,c) comes from the theory of cancellation of projective modules in the paper [69] of Swan–Towber where an explicit completion is stated in ([69], Theorem 2.1). Here are the two completions: Let $aa'+bb'+cc'=1$.

$$\begin{pmatrix} a^2 & b & c \\ b+ac' & -c'^2+ba'c' & -a'+b'c'-c'bb' \\ c-ab' & a'+b'c'+a'cc' & -b'^2-a'b'c \end{pmatrix}, \begin{pmatrix} a^2 & b & c \\ -b-2ac' & c'^2 & a'-b'c' \\ -c+2ab' & -a'-b'c' & b'^2 \end{pmatrix}.$$

Can the Swan-Towber method of computation be extended to give completions of the universal factorial row, in view of Suslin's theorem in [58]. Is there some interpretation of those completions akin to the theory which Suslin has built. (Note that both approaches are derived from an explicit computation to show the transitivity of the group of automorphisms of a projective module $P\oplus R$ on its unimodular elements.)

Let us commence on a different tack. Bass observed that the projective module $P_v=\ker(R^{2n}\stackrel{v}{\to}R)$ corresponding to a unimodular row $v=(v_1,v_2,\ldots,v_{2n})$ of even length always has a unimodular element, i.e. it splits of a free summand isomorphic to R: $w=(v_2,-v_1,v_4,-v_3,\ldots,-v_{2n},v_{2n-1})\in P_v$ and is a unimodular row.

Raja Sridharan and Ravi Rao observed that if $\chi_2(v)=(v_1^2,v_2,\ldots,v_{2n-1})\in \mathrm{Um}_{2n-1}(R)$ then the projective module $P_{\chi_2(v)}$ has a unimodular element. (See ([38], pg. 120, Theorem 5.6) for a more general statement).

S.M. Bhatwadekar commented on seeing this that a unimodular row of the form $(a_0^2,a_1,a_2,a_3^2,a_4,a_5)$ has two independent sections! T.Y. Lam (with inputs from R.G. Swan) also began the study of **Sectionable sequences** in ([38], §5, pg. 116) to make a preliminary study of this phenomenon.

Can one recover Suslin's theorem on the completion of the 'universal factorial unimodular row' by using such an argument? In particular, to begin with, can one show that a unimodular row of the form $(a_0^6,a_1,\ldots,a_{2n})$ has two independent sections? etc.

• Suslin used it **in the computation of K-theory and $\mathcal{K}$-cohomology of group varieties** SL_n, GL_n, Sp_{2n}, etc. in [66]. We refer the reader to [66] where Suslin showed that

$$\mathrm{SK}_1\left(\frac{\mathbb{Z}[x_1,\dots,x_n,y_1,\dots,y_n]}{(\sum_{i=1}^n x_iy_i - 1)}\right) \simeq \mathbb{Z},$$

with generator $[S_{n-1}((x_1,\dots,x_n),(y_1,\dots,y_n))]$. Is the group $\mathrm{SL}_n(A)/\mathrm{E}_n(A)$, for $A = \mathbb{Z}[x_1,\dots,x_n,y_1,\dots,y_n]/(\sum_{i=1}^n x_iy_i - 1)$ generated by $[\beta_{n-1}(v,w)]$, for $v,w \in Um_n(A)$, with $\langle v,w\rangle = 1$? (This may depend on n, but is it true atleast in the metastable range $n \le 2d-3$, where d is dimension of A?)

• **Patching information in set-theoretic complete intersection problems.**

M. Boratyński showed in [8] that an ideal I in a polynomial ring R over a field can be generated upto radical by $m = \mu(I/I^2)$ elements, i.e. $\sqrt{I} = \sqrt{(f_1,\dots,f_m)}$, for some $f_1,\dots,f_m \in R$.

This is the first recorded use of the matrices $\beta_r(v,w)$ in the subject of Serre's program, followed by the Eisenbud–Evans program, which bridges properties of projective modules over a ring and the efficient generation of ideals in that ring. It replaces the homological methods used by Serre, and later by others like N. Mohan Kumar, M.P. Murthy in this context. The book [22] gives a nice introduction and survey of major previous literature on this topic.

Let us quickly recall M. Boratyński's idea: He says that if $\{x_1,\dots,x_m\} \subset I$ with $\{\overline{x}_1,\dots,\overline{x}_m\}$ generating the R/I-module I/I^2, and if J is the ideal generated by $(x_1,x_2,x_3^2,\dots,x_m^{m-1})$, and $I^{(m-1)!}$, then $\sqrt{J} = \sqrt{I}$, and the projective R- module got by taking the fibre product

$$P = R_t^m \times_{\beta_{m-1}((x_1,\dots,x_m))} R_{1-t}^m$$

maps onto J, for any $t \in R$ with $(1-t)I \subset (x_1,\dots,x_m)$. (Such a t is readily found, and the fact that J is locally generated by the obvious m elements on the open set $D(1-t)$, and by one element on $D(t)$, is easily verified. This information is 'patched' via $\beta_{m-1}((x_1,\dots,x_m))$.

By the Quillen-Suslin theorem ([40], [60]) P is free, and so J is generated by m elements.

Thus, M. Boratyński encoded Quillen's idea of local patching to ideals, and pushed forward Serre's program of projective generation of ideals; via a compressed version of a Suslin matrix.

• **Defining higher Mennicke symbols on orbits of unimodular rows.**

R. Fossum, H. Foxby, B. Iversen defined, for $n \ge 2$, a Mennicke n-symbol $\mathrm{Um}_n(R) \overset{\mathrm{wt}}{\to} \mathrm{SK}_1 R$ using the theory of acyclic based complexes. (We refer the reader to [17]; a copy of which can be got by making a request.)

Let $v = (a_1,\dots,a_n)$, $w = (b_1,\dots,b_n) \in \mathrm{Um}_n(R)$, with $\langle v,w\rangle = v\cdot w^t = 1$. The Koszul complex

$$X(v) \quad = \quad (\ldots \to \wedge^k(R^n) \overset{d_v}{\to} \wedge^{k-1}(R^n) \to \ldots)$$

is an acyclic based complex, with each $X_k(v) = \wedge^k(R^n)$ a free module with a canonical basis of exterior products $e_{i_1} \wedge \ldots \wedge e_{i_k}$, ordered lexicographically. External multiplication by w defines a contraction, say β for $X(v)$.

Since $(d+\beta)^2 = 1 + \beta^2$, and β is nilpotent, we get an isomorphism, independent of choice of the contraction,

$$X(v)_{odd} = \oplus X_{2i-1}(v) \quad \longrightarrow \quad \oplus X_{2i}(v) = X(v)_{even}.$$

$$\mathrm{wt}(v) = (-1)^{\binom{2n-1}{n}}[d+\beta] \in \mathrm{SK}_1(R)$$

Suslin interprets this map in ([65], §2) and showed that

$$\mathrm{wt}(v) = [S_{n-1}(v, w)] \in \mathrm{SK}_1(R).$$

(The reader may consult [46] where details are worked out.)

• **Dual is not isomorphic**: Let $\sum_{i=1}^n x_i y_i = 1$. Let P be the projective module corresponding to the unimodular rows $(x_1, \ldots, x_n)$. Then the dual P^* of P, i.e. $\mathrm{Hom}_R(P, R)$, is isomorphic to the projective R-module corresponding to the unimodular row $w = (\overline{y}_1, \ldots, \overline{y}_n) = w$.

It can be seen easily that P and P^* are isomorphic when rank P is odd; in fact, the rows v, w are in the same elementary orbit by a lemma of M. Roitman in ([56], Lemma 1).

However, if $n > 1$ is odd then there are several approaches due to M.V. Nori, R.G. Swan, who have independently shown (using topological arguments) that P, P^* are not isomorphic. For an exposition of this see the homepage of R.G. Swan at [70], [71].

Together with these approaches, we gave an approach via Suslin matrices following an argument of Suslin in [65]. We refer the reader to [47] where some of the approaches are collated. We mention the approach via Suslin matrices below: Let

$$R = \frac{\mathbb{Z}[x_1, \ldots, x_{2n-1}, y_1, \ldots, y_{2n-1}]}{\left(\sum_{i=1}^{2n-1} x_i y_i - 1\right)}.$$

Suppose that $v\sigma = w$, for some $\sigma \in \mathrm{GL}_{2n-1}(R)$. Then

$$\mathrm{wt}\ (w) = \mathrm{wt}\ (v\sigma) = \mathrm{wt}\ (v) + \sum_{i=0}^{2n-1} (-1)^i [\wedge^i \sigma].$$

Since $\mathrm{SK}_1(R) = \mathbb{Z}$, $[\sigma] = [S_{2n-2}(v, w)]^r$, for some r. Hence, $[\wedge^i \sigma] = r[\wedge^i S_{2n-2}(v, w)]$. Therefore,

$$\begin{aligned} \sum_{i=0}^{2n-1} (-1)^i [\wedge^i \sigma] &= r \sum_{i=0}^{2n-1} (-1)^i [\wedge^i S_{2n-2}(v, w)] \\ &= r\ \mathrm{wt}(\overline{x}_1, \overline{x}_2, \overline{x}_3^{-2}, \ldots, \overline{x}_{2n-1}^{2n-2}) \\ &= r(2n-2)!\ \mathrm{wt}(v). \end{aligned}$$

Thus,

$$\mathrm{wt}(w) = [S_{2n-2}(w, v)] = (1 + r(2n-2)!)\mathrm{wt}(v) = (1 + r(2n-2)!)[S_{2n-2}(v, w)].$$

But since v is of odd length, $[S_{2n-2}(w, v) = [S_{2n-2}(w, v)^t] = [S_{2n-2}(w, v)]^t$, by the identities of Suslin (detailed a little later), and using the nomality of the elementary linear subgroup (see ([61], Corollary 1.4)). But $S_{2n-2}(v, w)S_{2n}(w, v)^t = I$, and so $[S_{2n-2}(v, w)] = [S_{2n-2}(w, v)]^{-1}$.

Thus, one gets $(2 + r(2n-2)!)\mathrm{wt}\ (v) = 0$. A contradiction except when $n = 2, r = -1$. $\square$

• The Suslin matrices can be used **to derive properties of the orbit space of unimodular rows**. Consider the following two principles:

- $\star$ (Generalized Local Global Principle): Let $v(X),\ w(X) \in \mathrm{Um}_r(R[X])$, $r \geq 3$. Suppose that $v(X)_{\mathfrak{p}} \in w(X)_{\mathfrak{p}}\mathrm{E}_r(R_{\mathfrak{p}}[X])$, for all $\mathfrak{p} \in$Spec(R), and $v(0) = w(0)$, then is $v(X) \in w(X)\mathrm{E}_r(R[X])$?
- $\star$ (Generalized Monic Inversion Principle): Let $v(X),\ w(X) \in \mathrm{Um}_r(R[X])$, $r \geq 3$. Let $f(X) \in R[X]$ be a monic polynomial. Suppose that $v(X)_{f(X)} \in w(X)_{f(X)}\mathrm{E}_r(R[X]_{f(X)})$, then is $v(X) \in w(X)\mathrm{E}_r(R[X])$?

Both the above questions were also raised by T.Y. Lam in ([38], Chapter VIII, 5.6, 5.11). We gave a partial answer in [46] where we showed that $\chi_2([v(X)]) = \chi_2([w(X)])$, if r is odd, and $\chi_4([v(X)]) = \chi_4([w(X)])$, if r is even. (Here if $v = (v_1, \ldots, v_r) \in \mathrm{Um}_r(R)$ then $\chi_n([v])$ denotes the class of the row $(v_1^n, \ldots, v_r)$ (under elementary column operations). This is shown to be well defined in [76] by L.N. Vaserstein.)

• The Suslin matrices have thus been found useful for the study of unimodular rows; which are associated to 1-stably free projective modules. Can such a similar study also be done for any stably free projective module.

It is natural to expect that an analogous Suslin theory will develop for a pair $(p, a) \in P \oplus R$, $(\psi, b) \in P^* \oplus R$, with $\psi(p) + ab = 1$.

• Suslin studied **the transitive action of the orthogonal group on rows of length one** in ([58], Lemma 5.4). The very existence of $S_3(v, w)$ implies that $\mathrm{O}_8(R)$ acts transitively on the set of rows of length one, i.e. $\{(v, w), v, w \in \mathrm{Um}_4(R), \langle v, w\rangle = 1\}$. In ([26], Corollary 4.5) we showed that $\mathrm{SO}_{2n}(R)$ acts transitively on pairs having the further property that

$$[v] = \begin{cases} \chi_2([v']] & \text{if } n \text{ is odd} \\ \chi_4([v']) & \text{if } n \text{ is even} \end{cases}$$

Consequently, in view of Lemma 25 which comes a little later, if R is an affine algebra of dimension d over a perfect C_1 field, or if $R = A[X]$, A a local ring in which 2 is invertible, in view of ([42], Theorem 1), then $\mathrm{SO}_{2(d+1)}(R)$ acts transitively on rows of length one.

• Bass–Milnor–Serre began the study of the stabilization for the linear group $\mathrm{GL}_n(R)/\mathrm{E}_n(R)$ for $n \geq 3$, where R is a commutative ring with identity. In [5], they showed that $\mathrm{K}_1(R) = \mathrm{GL}_{d+3}(R)/\mathrm{E}_{d+3}(R)$, where d is the dimension of the maximum spectrum. In [72], L.N. Vaserstein proved their conjectured bound of $(d+2)$ for an associative ring with identity, where d is the stable dimension of the ring. After that, in [73], he introduced the orthogonal and the unitary K_1-functors, and obtained stabilization theorems for them. He showed that the natural map

$$\begin{cases} \varphi_{n,n+1} : \frac{\mathrm{S}(n,R)}{\mathrm{E}(n,R)} \longrightarrow \frac{\mathrm{S}(n+1,R)}{\mathrm{E}(n+1,R)} & \text{in the linear case} \\ \varphi_{n,n+2} : \frac{\mathrm{S}(n,R)}{\mathrm{E}(n,R)} \longrightarrow \frac{\mathrm{S}(n+2,R)}{\mathrm{E}(n+2,R)} & \text{otherwise} \end{cases}$$

(where $\mathrm{S}(n,R)$ is the group of automorphisms of the projective, symplectic and orthogonal modules of rank n with determinant 1, and $\mathrm{E}(n,R)$ is the elementary subgroup in the respective cases) is surjective for $n \geq d+1$ in the linear case, for $n \geq d$ in the symplectic case, and for $n \geq 2d+2$ in the orthogonal case, and is injective for $n \geq 2d+4$ in the symplectic and the orthogonal cases. Soon after, in [75], he studied stabilization for groups of automorphisms of modules over rings and modules with quadratic forms over rings with involution, and obtained similar stabilization results.

The Suslin matrices have been **found useful in the study of injective stabilization for the $\mathbf{K}_1$-functor of the classical groups**:

Let A be a non-singular affine algebra of dimension $d > 1$ over a perfect C_1-field. In [50] it is shown that the natural map $\frac{\mathrm{SL}_n(A)}{\mathrm{E}_n(A)} \longrightarrow \frac{\mathrm{SL}_{n+1}(A)}{\mathrm{E}_{n+1}(A)}$ is injective for $n \geq d+1$. In [6] it is shown that if $(d+1)!A = A$, then the natural map $\frac{\mathrm{Sp}_n(A)}{\mathrm{E}_n(A)} \longrightarrow \frac{\mathrm{Sp}_{n+2}(A)}{\mathrm{ESp}_{n+2}(A)}$ is injective for $n \geq d+1$. Similar results have also been obtained in the case of the classical modules in [6]. The completion of the universal factorial row, and H. Lindel–T. Vorst results in [39], [77] on the Bass–Quillen conjecture, played a crucial role in proving these results.

In the symplectic situation, in [9] these results have been simplified to some extent using a relative version of Quillen's Local Global Principle in [1], coupled with the Suslin completions of the factorial row. It is shown in [9] that $v\mathrm{E}_{2n}(R,I) = v\mathrm{ESp}_{2n}(R,I)$, for any commutative ring R, and ideal I in R, and for any unimodular row $v \in \mathrm{Um}_n(R,I)$, $n \geq 3$. Using this one can recapture the earlier results; and also show that if R be a finitely generated algebra of even dimension d over K, where $K = \mathbb{Z}$ or a finite field or its algebraic closure, and if $\sigma \in \mathrm{Sp}_d(R)$ with $(I_2 \perp \sigma) \in \mathrm{ESp}_{d+2}(R)$, then σ is (symplectic) homotopic to the identity. In fact, $\sigma = \rho(1)$ for some $\rho(X) \in \mathrm{Sp}_d(R[X]) \cap \mathrm{ESp}_{d+2}(R[X])$, with $\rho(0) = I_d$. Finally, all these results were improved in [20]; and optimal bounds were obtained there for smooth algebras over an algebraically closed field by using the the Fasel–Rao–Swan theorem in [15]. Results of such type are also expected over a perfect field of cohomological dimension ≤ 1; but not over fields of cohomological dimension two, is demonstrated in [20], in view of N. Mohan Kumar's examples in [37] of non-free stably free modules of rank $d-1$ over a field of cohomological dimension 1.

The relative strengthening of L.N. Vaserstein's famous lemma (in [67]) that $e_1\mathrm{E}_{2n}(R) = e_1\mathrm{ESp}_{2n}(R)$ done in [9] can also be deduced from it and the Excision theorem of W. van der Kallen in ([31], Theorem 3.21), via the Key lemma for Suslin matrices. In fact, one can even get the stronger $v\mathrm{E}_{2n}(R) = v\mathrm{ESp}_{\varphi_{2n}}(R)$, for any unimodular row $v \in \mathrm{Um}_{2n}(R)$, and any invertible alternating matrix φ, for an appropriate definition of $\mathrm{ESp}_{\varphi_{2n}}(R)$. It is an instructive exercise for the reader to figure this out using the material in this text.

The study of injective stabilization is useful to answer a question of Suslin in [64] regarding whether a stably free projective module of rank $(d-1)$ over a (non-singular) affine algebra of dimension d over an algebraically closed field, with some divisibility conditions, is free. This will be true for even dimensions if the injective stability estimate for $\mathrm{K}_1\mathrm{Sp}$ falls to $d-1$, over odd dimensional (non-singular) affine algebras of dimension d over a perfect C_1-field. This will be true in any dimension if the injective stability for K_1 will fall to d over a d dimensional (non-singular) affine algebras over a perfect C_1-field.

The latter was established in [15]; but as a *consequence* of establishing Suslin's question for non-singular affine algebras over an algebraically closed field. (The contracted Suslin matrices played a vital role in its proof.)

• **The Suslin symbol**: In ([58], §5) introduced the groups $G_r(A)$. $G_r(A)$ is the Witt group of nonsingular quadratic forms if $r \equiv 0 \bmod 4$; $G_r(A)$ is the symplectic K_1 functor of the ring A if $r \equiv 1 \bmod 4$; $G_r(A)$ is the Witt group of nonsingular skew-symmetric forms if $r \equiv 2 \bmod 4$; $G_r(A)$ is the orthogonal K_1 functor of the ring A if $r \equiv 3 \bmod 4$.

One has the Suslin maps $S_r : \mathrm{Um}_{r+1}(A) \longrightarrow G_r(A)$ defined as follows: Choose a w such that $\langle v, w\rangle = 1$, and set

$$S_r(v) = \begin{cases} [S_r(v,w)] \text{ if } r = 2k+1 \\ [S_r(v,w) \cdot I_r] \text{ if } r = 2k. \end{cases}$$

For example, if $r = 1$ then the resulting map S_1 is precisely the well-known Mennicke symbol which had an important role in the solution of the congruence subgroup problem in [5]; for $r = 2$ S_2 is the Vaserstein symbol introduced in [67], and which was used to obtain some deep results on orbits of actions of $\mathrm{SL}_3(A)$ on $\mathrm{Um}_3(A)$. Suslin has asked for the meaning and properties of these maps. Our work in [26] was an initial attempt to understanding these maps and see if we could get some properties. We mention some progress on these questions below.

• **Hermitian K-theory**: One can reinterpret the groups $G_r(A)$ in the context of Hermitian K-theory as developed by M. Karoubi and, more recently, M. Schlichting. In [2], the authors show that these groups are avatars of higher Grothendieck-Witt groups. As said above, we have $G_1(A) = KSp_1(A)$ and $G_3(A) = KO_1(A)$. In Schlichting's notation, one writes $KSp_1(A) = GW_1^2(A)$ and $KO_1(A) = GW_1^0(A)$, where the letters GW stand for "Grothendieck-Witt" groups. These are bigraded abelian groups $GW_i^j(A)$ with $i \in \mathbb{Z}$ and $j \in \mathbb{Z}/4$. Suslin's symbol $Um_{r+1}(A) \to G_r(A)$ reads then as a collection of maps $Um_{r+1}(A) \to GW_1^{r+1}(A)$. In the same paper, A. Asok and J. Fasel show that Suslin's computation of the group SK_1 of the ring

$$A_n = \frac{\mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_n]}{(\sum_{i=1}^n x_i y_i - 1)}$$

refines in a computation of Grothendieck-Witt groups of A (with the price to consider $\mathbb{Z}[1/2]$-coefficients). Indeed, one finds

$$GW_1^{r+1}(A_{r+1}) = GW_0^0(\mathbb{Z}[1/2])$$

provided $r \geq 1$. There is an analogue of Quillen's spectral sequence computing K-theory in terms of codimension of the support in the theory of Grothendieck-Witt groups (see e.g. [14]). Asok and Fasel show that in the case of the ring A_{r+1}, an edge map in the corresponding spectral sequence is indeed an isomorphism. This allows to compute this edge map for any regular algebra (with 2 invertible) of dimension $\leq r$ in terms of Suslin matrices.

• **Study of orbit spaces, and classifying spaces**: If $R = C(X)$ is the ring of continuous real valued functions on a topological space X then every unimodular row $v \in \mathrm{Um}_n(C(X))$, $n \geq 2$, determines a map $arg(v) : X \longrightarrow \mathbb{R}^n - \{0\} \longrightarrow S^{n-1}$. (The first is by evaluation, and the second is the standard homotopy equivalence.) We thus get an element $[arg(v)]$ of $[X, S^{n-1}]$. (As $n \geq 2$, we may ignore base points.) Clearly, rows in the same elementary orbit define homotopic maps. Thus, we have a natural map $\mathrm{Um}_n(C(X))/\mathrm{E}_n(C(X)) \longrightarrow [X, S^{n-1}] = \pi^{n-1}(X)$.

Note that J.F. Adams has shown that S^{n-1} is not a H-space, unless $n = 1, 2, 4$, or 8. It is classically known that this is equivalent to saying that there is no suitable way to multiply the two projection maps $S^{n-1} \times S^{n-1}$ in $[S^{n-1} \times S^{n-1}, S^{n-1}]$. However, under suitable restrictions on the 'dimension' of X we may expect to define a product.

Henceforth, let X be a finite CW-complex of dimension $d \geq 2$. L.N. Vaserstein has shown that the ring $C(X)$ has stable dimension d. Now let $n \geq 3$, so that S^{n-1} will be atleast 1-connected. By the Suspension Theorem, the suspension map $S: \ [X; S^{n-1}] \longrightarrow [SX; S^n]$ is surjective if $d \leq 2(n-2)+1$, and bijective if $d \leq 2(n-2)$. Moreover, we know that $[SX, S^n]$ is an abelian group. Hence, the orbit space has a structure of an abelian group. It is shown in ([32], Theorem 7.7) that above map is a universal weak Mennicke symbol as defined by W. van der Kallen in [32].

In the context of commutative rings, for $n = 3$ and d atmost 2, the orbit space of unimodular rows modulo elementary action was shown to be bijective to the elementary symplectic Witt group (denoted by $\mathrm{W_E}(R)$) by L.N. Vaserstein in [67] and for $d \leq 2n - 4$, to the universal weak Mennicke symbol by W. van der Kallen in [32].

It would appear too strong to expect the bound to fall; and perhaps it is, but the article [45] encourages us, as it shows (using Suslin matrices) that there is a nice group structure on orbits of squares of unimodular rows when $\dim(R) \leq 2n-3$.

We say that the orbit space $\mathrm{Um}_r(R)/\mathrm{E}_r(R)$ has a Mennicke-like (or nice) structure if

$$[(a, a_2, \ldots, a_r)] \star [(b, a_2, \ldots, a_r)] = [(ab, a_2, \ldots, a_r)].$$

In ([18], Theorem 3.9) it is shown that if A is an affine algebra of dimension d over a perfect field k, of characteristic $\neq 2$, and with $\mathrm{c.d.}_2(k) \leq 1$, then if $r = d+1$, the van der Kallen group structure on it defined in [31] is Mennicke-like.

In [45] the Suslin matrix approach enables one to recapture this theorem when k is algebraically closed; and also to improve upon it for $r = d$, when k is a finite field. In fact, we realized later that the Suslin matrix approach in [45] would also enable us to recapture ([18], Theorem 3.9). We leave it to the reader to verify these details.

As pointed out in ([45], due to the strong results of J. Fasel in [16], for a smooth affine algebra over a field k, of characteristic $\neq 2$, and with $\mathrm{c.d.}_2(k) \leq 2$, the group structure on the orbit space $\mathrm{Um}_{d+1}(A)/\mathrm{E}_{d+1}(A)$ is nice. Is this the optimal situation for smooth affine algebras over a field?

The recent progress we have made is to relate these two studies, via the Suslin symbol. We briefly sketch this next.

• **Defining group structures, Witt group structures on orbits of unimodular rows**

One can define a Witt group $\mathrm{W}_{\mathrm{EUm}}(R)$, and a map from the orbit space $\mathrm{Um}_n(R)/\mathrm{E}_n(R) \longrightarrow \mathrm{W}_{\mathrm{EUm}(R)}$ sending $[v]$ to $[S_{n-1}(v, w)]$, for any w, with $\langle v, w\rangle = 1$. This map is a homomorphism, and is a Steinberg symbol if $\dim(R) \leq 2n-3$. It is also onto when $\dim(R) \leq 2n-3$. One can commence here as the variant of the Mennicke–Newmann lemma as in ([33], Lemma 3.2) is available. We expect it to also be injective under these conditions. This is mainly due to the inherent symmetry of the Suslin matrices.

Note that these would mean that the orbit space would then have a nice abelian Witt group structure under the condition $\dim(R) \leq 2n-3$; which is an improvement on the condition $\dim(R) \leq 2n-4$ in the theorem of van der Kallen in [32] stated above. More details will appear in [27], when n is even.

• In ([58], §3) Suslin points out that the fact that the universal factorial row can be completed can be used to find a completion of a linear unimodular row of length $(r+1)$, provided $r!$ is a unit. In fact he shows that there is a factorial row in the elementary orbit of any linear unimodular row. At the end of §5 he poses Problem 4 which reposes a question posed by Bass in [4], with an additional rider. We now know this as the Bass–Suslin conjecture; and it is one of the central open questions of classical algebraic K-theory. Let R be a local ring. Bass asked if $\mathrm{Um}_r(R[X]) = e_1\mathrm{SL}_r(R[X])$. Suslin expects this if $1/(r-1)! \in R$. More generally, due to Suslin's example, one would expect to find a factorial row in the elementary orbit of any unimodular row over a polynomial ring over a local ring.

The results of M. Roitman in [56], and R.A. Rao in [41],[42],[43] bear testimony to this. In [41] [42], [43] unimodular polynomial rows are studied via the Vaserstein symbol. In [27] a similar study is undertaken via the Suslin symbol. This study promises to solve this question in the metastable range; however, one expects that if one couples this with the ideas developing in [55] then one could get a complete picture, based on the beautiful symmetry of the Suslin matrices. More precisely, the structure of the Suslin matrix forces a certain positioning; and the argument in [27] indicates that some positionings (enforced by the positioning of the coordinates of a Suslin matrix) are suitable to enable us to lift the yoke of restriction of injective stability estimates of K_1 so far.

Historical development often gives a clue to the route one should follow.

The study of completions of unimodular rows over a commutative noetherian ring R of dimension d gives a hint of things to come. It began with J-P. Serre, followed by H. Bass, *ideas of general position*; which were taken further by Eisenbud–Evans. L.N. Vaserstein started studying group structures on orbits of unimodular rows *using Witt groups*. But the paper [67] already contains enough of *non-stable* algebraic K-theory arguments on a unimodular row; which were expanded upon by W. van der Kallen in [31], [32]. Thus, the arguments of [48] give preliminary historical evidence of getting completion of unimodular polynomial rows in dimension three by a stable argument. Injective stabilization plays an important role here; but we suspect that this happens because we have not done the linearization in a proper way which preserves the anti-symmetry.

It is this combination of ideas that **we strongly advocate** in the polynomial case; doing stable linearization, preserving the inherent symmetry of the Suslin matrices, and taking n-th roots, we believe should give a 'polynomial time' feedback completion algorithm at the non-stable level. We hope to be able to present these ideas in [55].

3 Study of the Suslin matrix

We begin with the study of the alternating matrices; which gives a good role model to begin the topic.

The alternating matrix $V(v, w)$

Let $v = (a, b, c)$, $w = (a', b', c')$ with $\langle v, w\rangle = aa' + bb' + cc' = 1$.

We consider the 4×4 alternating matrix $V(v, w)$ of Pfaffian one:

$$V(v,w) \quad = \quad \begin{pmatrix} 0 & a & b & c \\ -a & 0 & c' & -b' \\ -b & -c' & 0 & a' \\ -c & b' & -a' & 0 \end{pmatrix}$$

We wish to analyze the action of $\varepsilon \in \mathrm{E}_4(R)$ on $V(v, w)$ by conjugation.

We first recall **the Cohn transformations** of a row below:

Definition 1 *Let $v = (a_0, a_1, \ldots, a_r)$, $w = (b_0, b_1, \ldots, b_r) \in R^{r+1}$ with $\langle v, w\rangle = 1$. We say that the row*

$$v^* = vC_{ij}(\lambda) = (a_0, \ldots, a_i + \lambda b_j, \ldots, a_j - \lambda b_i, \ldots, a_r),$$

for $0 \le i \ne j \le r$, is a **Cohn transform** *of v w.r.t. the row w.*

P.M. Cohn in [13] had shown that the matrices $I_2 + \lambda \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} b & -a \end{pmatrix}$ were not elementary matrices in general.

It was shown in ([24], Lemma 2.1) that the Cohn orbit (got by a finite number of successive Cohn transforms) is the same as the elementary orbit when $r \ge 2$. Moreover, see ([34] Theorem 3.6), if $\langle v, w\rangle = \langle v', w\rangle = 1$ then v' can be got from v by a finite number of Cohn transforms w.r.t. w.

Let us get back to analysing the action of an elementary metrix on an alternating matrix.

One has the following identities:

$$\begin{aligned}
E_{12}(\lambda)V(v,w)E_{12}(\lambda)^t &= V(vC_{12}(\lambda), w),\\
E_{13}(\lambda)V(v,w)E_{13}(\lambda)^t &= V(vC_{02}(-\lambda), w),\\
E_{14}(\lambda)V(v,w)E_{14}(\lambda)^t &= V(vC_{01}(\lambda), w),\\
E_{21}(\lambda)V(v,w)E_{21}(\lambda)^t &= V(v, wC_{21}(\lambda)),\\
E_{31}(\lambda)V(v,w)E_{31}(\lambda)^t &= V(v, wC_{20}(-\lambda)),\\
E_{41}(\lambda)V(v,w)E_{41}(\lambda)^t &= V(v, wC_{10}(\lambda)).
\end{aligned} \tag{1}$$

Equations (1) describes completely the action of $\mathrm{E}_4(R)$ on an alternating matrix $V(v, w)$.

We may consider the Vaserstein space V of dimension 6 consisting of all 4×4 alternating matrices over R. The above relations associates a linear transformation T_σ of V with any $\sigma \in \mathrm{SL}_4(R)$ by $T_\sigma(V(v, w)) = \sigma V(v, w)\sigma^t$. The matrix of this linear transformation w.r.t the usual ordered basis $e_1, \ldots, e_6$ is not orthogonal. However, with respect to the following permutation of the standard basis $e_1, \ldots, e_6$ namely $e_1, e_2, e_3, e_6, -e_5, e_4$ we get

$$\begin{array}{ll}
E_{12}(x) \to E_{62}(x)E_{53}(-x) & E_{21}(x) \to E_{26}(x)E_{35}(-x)\\
E_{13}(x) \to E_{61}(-x)E_{43}(x) & E_{31}(x) \to E_{16}(x)E_{34}(-x)\\
E_{14}(x) \to E_{51}(-x)E_{42}(x) & E_{41}(x) \to E_{15}(x)E_{24}(x)
\end{array}$$

The images are all elementary orthogonal matrices. In particular, the matrix of T_σ will be an orthogonal matrix. One observes also that the map $\mathrm{E}_4(R)$ is onto $\mathrm{EO}_4(R)$. This induces an injection of the quotient groups $\mathrm{SL}_4(R)/\mathrm{E}_4(R) \longrightarrow \mathrm{SO}_4(R)/\mathrm{EO}_4(R)$.

Let us compute T_σ. It is the matrix of $\wedge^2\sigma$. When $\sigma = V(v,w)$ something interesting is revealed: the matrix is $(I_4 - \begin{pmatrix} v^t \\ w^t \end{pmatrix} \begin{pmatrix} w & v \end{pmatrix})(I_4 - \begin{pmatrix} e_1^t \\ e_1^t \end{pmatrix} \begin{pmatrix} e_1 & e_1 \end{pmatrix})$. This is recognizable as the product of two reflections $\tau_{(v,w)} \circ \tau_{(e_1,e_1)}$. (See later for the definition.)

Is there a similar 'larger sizes' analogue? The observations above are replicated below with the Suslin matrix substituting for the alternating matrix $V(v,w)$.

Remark: When we did calculations with 6×6 alternating matrices of Pfaffian one we found that the corresponding linear transformations were not orthogonal, and so the theory is dissimilar. It seems worthwhile to investigate what is happening here.

The Suslin matrix $S_r(v,w)$

We now describe the Suslin matrices in more detail.

The construction of the Suslin matrix $S_r(v,w)$ is possible once we have two rows v, w. These matrices will be invertible if their dot product $v\cdot w^t = 1$. (The rows are then automatically unimodular rows.) Suslin's inductive definition: Let

$$v \quad = \quad (a_0, a_1, \ldots, a_r) = (a_0, v_1),$$

with $v_1 = (a_1, \ldots, a_r)$,

$$w \quad = \quad (b_0, b_1, \ldots, b_r) = (b_0, w_1),$$

with $w_1 = (b_1, \ldots, b_r)$. Set $S_0(v,w) = a_0$, and set

$$S_r(v,w) \quad = \quad \begin{pmatrix} a_0 I_{2^{r-1}} & S_{r-1}(v_1,w_1) \\ -S_{r-1}(w_1,v_1)^t & b_0 I_{2^{r-1}} \end{pmatrix}.$$

Suslin noted that $S_r(v,w)S_r(w,v)^t = (v\cdot w^t)I_{2^r} = S_r(w,v)^t S_r(v,w)$, and $\det S_r(v,w) = (v\cdot w^t),^{2^{r-1}}$ for $r \geq 1$.

Thus the positions of a_i and b_i in $S_r(v,w)$ as follows: For $1 \leq i \leq r-1$,

1. The positions of a_0 in $S_r(v,w)$ is given by (k,k), $1 \leq k \leq 2^{r-1}_,$ and the positions of b_0 in $S_r(v,w)$ is given by (k,k), $2^{r-1}+1 \leq k \leq 2^r_.$
2. The positions of a_r in $S_r(v,w)$ is given by $(2k-1, 2^r-2k+2), 1 \leq k \leq 2^{r-1}_,$ and the positions of b_r in $S_r(v,w)$ is given by $(2k, 2^r-2k+1), 1 \leq k \leq 2^{r-1}_.$
3. The positions of $+a_i$ in $S_r(v,w)$ is given by

   $$(2^2k2^{r-1-i}+j, (2+(2^{i-1}-k-1)2^2)2^{r-1-i}+j),$$

   where $0 \leq k \leq 2^{i-1}-1$, $1 \leq j \leq 2^{r-1-i}_.$

4. The positions of $-a_i$ in $S_r(v,w)$ is given by

$$((3+2^2k)2^{r-1-i}+j,(1+(2^{i-1}-k-1)2^2)2^{r-1-i}+j),$$

where $0 \le k \le 2^{i-1}-1$, $1 \le j \le 2^{r-1-i}$

5. The positions of $+b_i$ in $S_r(v,w)$ is given by

$$((1+(2^{i-1}-k-1)2^2)2^{r-1-i}+j,(3+2^2k)2^{r-1-i}+j),$$

where $0 \le k \le 2^{i-1}-1$, $1 \le j \le 2^{r-1-i}$

6. The positions of $-b_i$ in $S_r(v,w)$ is given by

$$(2+(2^{i-1}-k-1)2^2)2^{r-1-i}+j,2^2k2^{r-1-i}+j),$$

where $0 \le k \le 2^{i-1}-1$, $1 \le j \le 2^{r-1-i}$

The Suslin forms J_r

To understand the nature of the shape of the Suslin matrices we recall Suslin's sequence of forms $J_r \in \mathrm{M}_{2^r}(R)$ given by the recurrence formulae:

$$J_r = \begin{cases} 1 & \text{for r} = 0 \\ J_{r-1} \perp -J_{r-1}, & \text{for r even }, \\ J_{r-1} \top - J_{r-1}, & \text{for r odd.} \end{cases}$$

(The English translation wrongly says $J_r = J_{r-1} \perp J_{r-1}$ when r is even.)

(Here $\alpha \perp \beta = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, while $\alpha \top \beta = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}$.)

How did Suslin think of these forms? What will the form be if the 'Suslin matrix' is constructed by a slightly different basis; say by the usual lexicographic ordering of the basis to describe the map $\oplus_{i\text{odd}} \wedge^i R^r \to \oplus_{i\text{even}} \wedge^i R^r$ in the earlier construction. We give a possible approach: Observe that $J_r = \prod_{i=1}^{r+1} S_r(e_i, e_i)$. The reader can verify this by an easy induction on r. (Or can refer to [46] where it is proved.)

It is easy to see that $\det J_r = 1$, for all r, and that $J_r^t = J_r^{-1} = (-1)^{\frac{r(r+1)}{2}} J_r$. Moreover, J_r is antisymmetric if $r = 4k+1$ and $r = 4k+2$, whereas J_r is symmetric for $r = 4k$ and $r = 4k+3$.

We know from Suslin that he was unaware of M. Krusemeyer's explanations in [35] [36] for the Swan-Towber completion of (a^2, b, c). The explanations of M. Krusemeyer seem to be adequate only in the case of alternating forms. (Are we wrong in saying this?)

Suslin recognized the need to analyse the shapes of the Suslin matrices $S_r(v,w)$. He realized that the shapes satisfied similar properties according to the length $(r+1)$ of the row.

In ([58], Lemma 5.3), it is noted that the following formulae are valid:

$$\begin{array}{rcl}
\text{for } r=4k: (S_r(v,w)J_r)^t &=& S_r(v,w)J_r;\\
\text{for } r=4k+1: S_r(v,w)J_rS_r(v,w)^t &=& (v\cdot w^t)J_r;\\
\text{for } r=4k+2: (S_r(v,w)J_r)^t &=& -S_r(v,w)J_r;\\
\text{for } r=4k+3: S_r(v,w)J_rS_r(v,w)^t &=& (v\cdot w^t)J_r.
\end{array}$$

We call these **the Suslin identities**. These identities are the core of the underlying four physical configuration spaces in which unimodular rows live.

These identities may be easily verified by induction on r. Alternatively, one can also observe it after noting that for $r \geq 1$, and $2 \leq i \leq r+1$, $S_r(e_i,e_i)^{-1} = S_r(e_i,e_i)^t = -S_r(e_i,e_i)$, $S_r(e_i,e_i)^2 = -I_{2^r}$, and $\det S_r(e_i,e_i) = 1$, and the following lemma:

Lemma 2 *Let $v = (a_0,a_1,\ldots,a_r)$, $w = (b_0,b_1,\ldots,b_r) \in \mathrm{M}_{1r+1}(R), r \geq 1$. Then for $2 \leq i \leq r+1$,*

$$S_r(e_i,e_i)S_r(v,w)S_r(e_i,e_i)^{-1} = S_r(v',w'),$$

where

$$\begin{array}{rcl}
v' &=& (b_0,-a_1,\ldots,-a_{i-2},b_{i-1},-a_i,\ldots,-a_r), \text{ and}\\
w' &=& (a_0,-b_1,\ldots,-b_{i-2},a_{i-1},-b_i,\ldots,-b_r).
\end{array}$$

Thus, one has

$$J_rS_r(v,w)J_r^{-1} = \begin{cases} S_r(v,w)^t & \text{if } r \text{ even}\\ S_r(w,v) & \text{if } r \text{ odd.}\end{cases}$$

The Suslin identities show that unimodular rows of length $r+1$ will have properties depending on $[r]$ modulo 4. We have already seen an instance of a property which depends on the parity of r when discussing the isomorphism of a projective module corresponding to a row and its dual projective module. Is there such an example of a property for unimodular rows which depends on the $[r]$ modulo 4?

When searching for an algorithm to create a $\beta_r(v,w)$ from $S_r(v,w)$ one should also keep the following question in mind. One knows that there is a $\beta_r(v,w) \in S_r(v,w)\mathrm{E}_{2^r}(R)$. When r is odd, is there a $\beta_r(v,w) \in S_r(v,w)\mathrm{GE}_{2^r}(R)$, where GE = ESp when $r = 4k+1$, and GE = EO when $r = 4k+3$? (The "right" ESp, EO is part of the query.)

The Fundamental property and the Key Lemma

We give a simple proof of the Fundamental property of Suslin matrices, which first appeared in [24].

Lemma 3 *Let R be a ring with 1. Let S be a subset of R satisfying*

1. *$a \in S$ implies $-a \in S$.*

2. $a,\ b \in S$ *implies* $a + b \in S$.

3. $a \in S$ *implies* $a^2 \in S$.

Then $a,\ b \in S$ *implies* $ab + ba \in S$, $2abc \in S$.

Proof: $ab + ba = \{(a+b)^2 - a^2\} - b^2 \in S$. Hence,

$$\{a(ab+ba) + (ab+ba)a\} - (a^2b + ba^2) = 2aba \in S.$$

We now state and prove the important Fundamental property satisfied by the Suslin matrices.

Corollary 4 (Fundamental property) *Let* $S_r(s,t)$, $S_r(v,w)$ *be Suslin matrices. Then*

$$\begin{aligned} S_r(s,t)S_r(v,w)S_r(s,t) &= S_r(v',w') \\ S_r(t,s)S_r(w,v)S_r(t,s) &= S_r(w',v'), \end{aligned}$$

for some $v', w' \in M_{1,r+1}(R)$, *which depend linearly on* v, w *and quadratically on* s, t. *Consequently,* $v' \cdot w'^{t} = (s \cdot t^t)^2 (v \cdot w^t)$.

Proof: Take $R = \mathrm{M}_{2^r}(R)$, and let S be the subset of all Suslin matrices above. Take $a = S_r(s,t)$, $b = S_r(v,w)$. Then $2aba \in S$. A generic argument will enable us to assume that 2 is a non-zero-divisor, and allow us to conclude that $aba \in S$.

The last two assertions will need the more specific argument given in ([26], Lemma 2.5).

Remark 5 *L. Avramov had independently observed a similar argument to prove the Fundamental property of Suslin matrices.*

The Key Lemma

Recall that we were led to the above Fundamental property in ([24], Corollary 3.3) by the Key Lemma via the methods of commutative algebra. We next recall the Key Lemma which is actually equivalent to the Fundamental Property. (We refer the reader to the thesis of Selby Jose ([23], Chapter 4, Lemma 4.3.16) where this equivalence has been detailed).

The Cohn transforms were first sighted in the work of L.N. Vaserstein in [67] when he considered the action of an elementary matrix on a 4×4 invertible alternating matrix as described earlier. His analysis led us to the key lemma below:

Notation. For a matrix $\alpha \in \mathrm{M}_k(R)$, we define α^{top} as the matrix whose entries are the same as that of α above the diagonal, and on the diagonal, and is zero below the diagonal. Similarly, we define α^{bot}.

For simplicity we may write α^t for α^{top}, α^b for α^{bot}, and α^T for α transpose. Moreover, we use α^{tb} for α^{top} or α^{bot}.

Lemma 6 (Key Lemma)
Let $v,\ w \in \mathrm{M}_{1,r+1}(R)$. Then, for $r \geq 2,\ 2 \leq i \leq r+1,\ \lambda \in R$,

$$\begin{aligned}
S_r(vE_{i1}(-\lambda), wE_{1i}(\lambda)) &= S_r(e_1, e_1E_{1i}(\lambda))^{top} S_r(v,w) S_r(e_1, e_1E_{1i}(\lambda))^{bot}_{,} \\
S_r(vE_{1i}(\lambda), wE_{i1}(-\lambda)) &= S_r(e_1E_{1i}(\lambda), e_1)^{bot} S_r(v,w) S_r(e_1E_{1i}(\lambda), e_1)^{top}_{,} \\
S_r(vC_{0i-1}(-\lambda), w) &= S_r(e_1E_{1i}(\lambda), e_1)^{top} S_r(v,w) S_r(e_1E_{1i}(\lambda), e_1)^{bot}_{,} \\
S_r(v, wC_{0i-1}(-\lambda)) &= S_r(e_1, e_1E_{1i}(\lambda))^{bot} S_r(v,w) S_r(e_1, e_1E_{1i}(\lambda))^{top}_{.}
\end{aligned}$$

In view of its own importance we wish to record the useful observation which led us to the proof of the Key Lemma given in ([24], Lemma 3.1):

Lemma 7 *Let $v,\ w,\ s,\ t \in R^{r+1}$ and let $v = (a_0, a_1, \ldots, a_r)$, $w = (b_0, b_1, \ldots, b_r)$. Then*

$$\begin{aligned}
S_r(v,w) + S_r(w,v)^t &= \{a_0 + b_0\} I_{2^r}. \\
S_r(s,t)S_r(w,v)^t + S_r(v,w)S_r(t,s)^t &= \{\langle s,w\rangle + \langle v,t\rangle\} I_{2^r}. \\
S_r(w,v)^t S_r(s,t) + S_r(t,s)^t S_r(v,w) &= \{\langle s,w\rangle + \langle v,t\rangle\} I_{2^r}.
\end{aligned}$$

Proof: This is the usual bilinear consequence of the quadratic relation

$$\begin{aligned}
S_r(v+s, w+t) S_r(w+t, v+s)^t &= \langle v+s, w+t\rangle I_{2^r} \\
&= \{\langle v,w\rangle + \langle v,t\rangle + \langle s,w\rangle + \langle s,t\rangle\} I_{2^r}.
\end{aligned}$$

The relations above reminds one of the relations in a Clifford algebra.

Commutator Calculus

Finally, we record a few interesting relations we got in ([24], Lemma 3.6) by use of the Key Lemma: This is the Yoga of commutators in the elementary unimodular vector group. As is known, a proper handle of this, can lead one to understand the quotient group of the Suslin unimodular vector group by its elementary unimodular vector group, better. In fact, it is eventually shown that this is a solvable group, using the methods of A. Bak in [3]. (See below for the definitions, and indications of a proof.)

Lemma 8 *Let $2 \leq i \neq j \leq r+1$, and let $\lambda = -2xy$. If*

$$\begin{aligned}
\alpha &= [S_r(e_1E_{1i}(x), e_1), S_r(e_1E_{1j}(y), e_1)], \\
\alpha^* &= [S_r(e_1E_{1j}(-y), e_1), S_r(e_1E_{1i}(-x), e_1)]
\end{aligned}$$

then $\alpha^* = \alpha^{-1}$, and $S_r(vC_{i-1j-1}(\lambda), w) = \alpha S_r(v,w)\alpha^{-1}$;

$$\begin{aligned}
\beta &= [S_r(e_1, e_1E_{1i}(x)), S_r(e_1, e_1E_{1j}(y))], \\
\beta^* &= [S_r(e_1E_{1j}(-y), e_1), S_r(e_1, e_1E_{1i}(-x))],
\end{aligned}$$

then $\beta^* = \beta^{-1}$, and $S_r(v, wC_{i-1j-1}(\lambda)) = \beta S_r(v,w)\beta^{-1}$;

$$\begin{aligned}
\gamma &= [S_r(e_1E_{1j}(x), e_1), S_r(e_1, e_1E_{1i}(y))], \\
\gamma^* &= [S_r(e_1, e_1E_{1i}(-y)), S_r(e_1E_{1j}(-x), e_1)],
\end{aligned}$$

then $\gamma^* = \gamma^{-1}$, and $S_r(vE_{ij}(\lambda), wE_{ji}(-\lambda)) = \gamma S_r(v,w)\gamma.^{-1}$

The Suslin Vector space

It is easy to see that the set

$$S = \{S_r(v,w) | v, w \in M_{1r+1}(R)\}$$

is a free R-module or rank $2(r+1)$. For a basis one can take $se_0, \ldots, se_{r+1}$, $se_0^*, \ldots, se_{r+1}^*$, where $se_i = S_r(e_i, 0)$, $se_i^* = S_r(0, e_i)$, for $0 \le i \le r$. We shall call this the **Suslin space**.

The Suslin Matrix Groups

Definition: The *Special Unimodular Vector group* $\mathrm{SUm}_r(R)$ is the subgroup of $\mathrm{SL}_{2^r}(R)$ generated by the Suslin matrices $S_r(v,w)$ w.r.t. the pair (v,w), with $v \in \mathrm{Um}_{r+1}(R)$, for some w with $\langle v, w\rangle = v \cdot w^t = 1$.

Remark 9 One can analogous to the linear case, define the Elementary Unimodular vector subgroup $\mathrm{EUm}_r(R)$ of $\mathrm{SUm}_r(R)$ generated by the Suslin matrices $S_r(v,w)$, with $v = e_1\varepsilon$, for some $\varepsilon \in \mathrm{E}_{r+1}(R)$, and with $v \cdot w^t = 1$.

Proposition 10 (Center of $\mathrm{SUm}_r(R)$) ([26], Corollary 3.5)
Let R be a commutative ring. The center $Z(\mathrm{SUm}_r(R))$ of the Special Unimodular vector group $SUm_r(R)$ consists of scalar matrices uI_{2^r}. Moreover,

$$Z(\mathrm{SUm}_r(R)) = \begin{cases} \{uI_{2^r} : u \in R, u^2 = 1\}, & \text{if } r \text{ odd} \\ \{uI_{2^r} : u \in R, u^4 = 1\}, & \text{if } r \text{ even.} \end{cases}$$

Hence $Z(\mathrm{SUm}_r(R)) \subseteq \mathrm{EUm}_r(R)$.

Commutator Calculus (contd.)

There is yet another set of generators for $\mathrm{EUm}_r(R)$, *viz.* $S_r(e_1E_{1i}(x), e_1)$, $S_r(e_1, e_1E_{1i}(y))$, and $S_r(e_i, e_iE_{i1}(a))$, $S_r(e_iE_{i1}(b), e_i)$, for $2 \le i \le r+1$, $x, y, a, b \in R$. This was shown in [24], via the Key Lemma 6 and Lemma 13.

We record the commutator formulae in $\mathrm{EUm}_r(R)^{tb}$ next: We use the convenient notation that for $r \ge 1$, $1 \le i \le r+1$, $\lambda \in R$,

$$\begin{aligned} E(e_1)(\lambda) &= I_{2^r} = E(e_1^*)(\lambda) \\ E(e_i)(\lambda) &= S_r(e_1E_{1i}(\lambda), e_1);\ i > 1 \\ E(e_i^*)(\lambda) &= S_r(e_1, e_1E_{1i}(\lambda));\ i > 1 \end{aligned}$$

(If we wish to stress the size we will write $E_r(e_i)(\lambda)$, $E_r(e_i^*)(\lambda)$). [1]

[1] The definition of $E(e_1)(\lambda)$ was erroneously defined as $\lambda I_{2^{r-1}} \perp \lambda^{-1} I_{2^{r-1}}$ in [24].

Proposition 11 *For $r \geq 2$, $\lambda, \mu \in R$, $c_i = e_i$ or e_i^*, $d_j = e_j$ or e_j^*, we have, for $2 \leq i < j \leq r+1$,*

$$\begin{aligned}
&[E_r(c_i)(\lambda)^t, E_r(d_j)(\mu)^b] \\
&\quad = \quad [E_{r-1}(c_{i-1})(\lambda)^t, E_{r-1}(d_{j-1})(\mu)^b] \perp [E_{r-1}(c_{i-1})(\lambda)^t, E_{r-1}(d_{j-1})(\mu)^b] \\
&\quad = \quad \underbrace{\alpha \perp \cdots \perp \alpha}_{2^{i-2} \text{ times}},
\end{aligned}$$

where

$$\alpha = \begin{cases} \{E_{r-i+1}(d_{j-i+1})(\lambda\mu)^{top} \perp E_{r-i+1}(d_{j-i+1})(-\lambda\mu)^{bot}\} & \text{if } c_i = e_i, \\ \{E_{r-i+1}(d_{j-i+1})(\lambda\mu)^{bot} \perp E_{r-i+1}(d_{j-i+1})(-\lambda\mu)^{top}\} & \text{if } c_i = e_i^*. \end{cases}$$

We next calculate the triple commutators:

Lemma 12 *For $r \geq 2$, $2 \leq i \neq j \leq r+1$, $\lambda, \mu, \nu \in R$,*

(i) $\big[[E(e_i)(\lambda)^{top}, E(e_j)(\mu)^{bot}], E(e_i^*)(\nu)^{tb}\big] = E(e_j)(\lambda\mu\nu)^{tb}_{,}$

(ii) $\big[[E(e_i^*)(\lambda)^{top}, E(e_j)(\mu)^{bot}], E(e_i)(\nu)^{tb}\big] = E(e_j)(\lambda\mu\nu)^{tb}_{,}$

(iii) $\big[[E(e_i)(\lambda)^{top}, E(e_j^*)(\mu)^{bot}], E(e_i^*)(\nu)^{tb}\big] = E(e_j^*)(\lambda\mu\nu)^{tb}_{,}$

(iv) $\big[[E(e_i^*)(\lambda)^{top}, E(e_j^*)(\mu)^{bot}], E(e_i)(\nu)^{tb}\big] = E(e_j^*)(\lambda\mu\nu)^{tb}_{.}$

The Key Lemma makes us consider the subgroup $\mathrm{EUm}_r(R)^{tb}$ of $\mathrm{E}_{2^r}(R)$ generated by elements of the type $S_r(e_1E_{1i}(x), e_1)^{tb}$, $S_r(e_1, e_1E_{1i}(x))^{tb}$. In view of the Key Lemma it is clear that $\mathrm{EUm}_r(R) \subset \mathrm{EUm}_r(R)^{tb}$.

Via the triple commutator laws, one gets the following relations, which prove the fact that $\mathrm{EUm}_r(R) = \mathrm{EUm}_r(R)^{tb}$.

Lemma 13 ([26], Lemma 4.9)
For $r \geq 2$, $2 \leq i \neq j \leq r+1$, and $\lambda \in R$.

$$\begin{aligned}
S_r(e_1E_{1i}(\lambda), e_1)^{top} \quad = \quad & S_r(e_1E_{1j}(\lambda), e_1)S_r(e_1 - \lambda e_j + e_i, e_1)S_r(e_1E_{1i}(-1), e_1) \\
& S_r((1-\lambda)e_1 + \lambda e_j + e_i, e_1 + e_j)S_r(e_1 - e_i, e_1 - e_j) \\
& S_r((1+\lambda)e_1 - \lambda e_j, e_1 + e_j)S_r(e_1, e_1E_{1j}(-1)), \\
S_r(e_1E_{1i}(\lambda), e_1)^{bot} \quad = \quad & S_r(e_1, e_1E_{1j}(-1))S_r((1+\lambda)e_1 - \lambda e_j, e_1 + e_j) \\
& S_r(e_1 - e_i, e_1 - e_j)S_r((1-\lambda)e_1 + \lambda e_j + e_i, e_1 + e_j) \\
& S_r(e_1E_{1i}(-1), e_1)S_r(e_1 - \lambda e_j + e_i, e_1)S_r(e_1E_{1j}(\lambda), e_1) \\
S_r(e_1, e_1E_{1i}(\lambda))^{top} \quad = \quad & S_r(e_1E_{1j}(-1), e_1)S_r(e_1 + e_j, (1+\lambda)e_1 - \lambda e_j) \\
& S_r(e_1 - e_j, e_1 - e_i)S_r(e_1 + e_j, (1-\lambda)e_1 + \lambda e_j + e_i) \\
& S_r(e_1, e_1E_{1i}(-1))S_r(e_1, e_1 - \lambda e_j + e_i)S_r(e_1, e_1E_{1j}(\lambda)), \\
S_r(e_1, e_1E_{1i}(\lambda))^{bot} \quad = \quad & S_r(e_1, e_1E_{1j}(\lambda))S_r(e_1, e_1 - \lambda e_j + e_i)S_r(e_1, e_1E_{1i}(-1)) \\
& S_r(e_1 + e_j, (1-\lambda)e_1 + \lambda e_j + e_i)S_r(e_1 - e_j, e_1 - e_i) \\
& S_r(e_1 + e_j, (1+\lambda)e_1 - \lambda e_j)S_r(e_1E_{1j}(-1), e_1)
\end{aligned}$$

(*Note that the alternate relations are got by reversing the order*).

The first step in computing the center $Z(\mathrm{SUm}_r(R))$ is to show that it consists of scalars. We give a different proof than in [24] of the fact that $Z(\mathrm{SUm}_r(R))$ consists of scalar matrices. We use the fact here that $\mathrm{EUm}_r(R)^{tb} = \mathrm{EUm}_r(R)$, for $r > 1$.

Lemma 14 *Let $A \in \mathrm{M}_{2^s}(\mathrm{M}_{2^t}(R))$, $t \geq 1$, $s + t = r$ be a diagonal block matrix, where the alternating diagonal blocks are the same. If A commutes with $E_r(e_{s+1})(1)^{top}$ and $E_r(e^*_{s+1})(1)^{top}_,$ then $A \in \mathrm{M}_{2^{s+1}}(\mathrm{M}_{2^{t-1}}(R))$ is a diagonal block matrix whose alternating diagonal block entries are same.*

Proof: Let $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $\begin{pmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{pmatrix} \in \mathrm{M}_{2^{t-1}}(\mathrm{M}_2(R))$ be the two, perhaps different, diagonal blocks of A. Compare the $(1, 2^s)$-th, $(1, 2^s-1)$-th, and $(2, 2^s-1)$-th block entries of $AE_r(e_{s+1})(1)^{top}$ and $E_r(e_{s+1})(1)^{top}A$ we get $a_{21} = 0$, $a_{34} = 0$, and $a_{11} = a_{33}$ respectively. Compare the $(1, 2^s)$-th, $(2, 2^s)$-th, and $(2, 2^s - 1)$-th block entries of $AE_r(e^*_{s+1})(1)^{top}$ and $E_r(e^*_{s+1})(1)^{top}A$ we get $a_{12} = 0$, $a_{22} = a_{44}$, and $a_{43} = 0$ respectively. Hence $A \in \mathrm{M}_{2^{s+1}}(\mathrm{M}_{2^{t-1}}(R))$ and is a diagonal matrix with alternating entries equal.

Lemma 15 *Let $A \in \mathrm{M}_{2^r}(R)$ be a diagonal matrix with equal alternating diagonal entries. If A commutes with $E_r(e_{r+1})(1)^{top}_,$ then A is a scalar matrix.*

Proof: Let a_{11} and a_{22} be the two different diagonal entries of the matrix A. Compare the $(1, 2^r)$-th entry of $AE_r(e_{r+1})(1)^{top}$ and $E_r(e_{r+1})(1)^{top}A$, we get $a_{11} = a_{22}$. Hence A is a scalar matrix.

Proposition 16 (Center of $\mathrm{SUm}_r(R)$)
Let $A \in \mathrm{M}_{2^r}(R)$. If A commutes with every element of $\mathrm{SUm}_r(R)$, then A is a scalar matrix.

Proof: Since $\mathrm{SUm}_2(R) = \mathrm{Sl}_2(R)$, the result is clear for $r = 1$. So let $r \geq 2$. Let us write $A = (a_{ij})_{1 \leq i,j \leq 4}$ in block form. By comparing entries we observe that

(1) $E_r(e^*_2)(1)^{top}A = AE_r(e^*_2)(1)^{top}$ implies $a_{12} = a_{32} = a_{41} = a_{42} = a_{43} = 0$, $a_{22} = a_{44}$,

(2) $E_r(e_2)(1)^{top}A = AE_r(e_2)(1)^{top}$ implies $a_{21} = a_{31} = a_{34} = 0$, $a_{11} = a_{33}$,

(3) $E_r(e^*_2)(1)^{bot}A = AE_r(e^*_2)(1)^{bot}$ implies $a_{13} = a_{14} = a_{23} = 0$, and

(4) $E_r(e_2)(1)^{bot}A = AE_r(e_2)(1)^{bot}$ implies $a_{24} = 0$.

Hence $A \in \mathrm{M}_{2^2}(\mathrm{M}_{2^{r-2}}(R))$ is a diagonal block matrix with alternating diagonal blocks same. Apply Lemma 14 $r - 2$ times and conclude that A is a diagonal matrix with alternating entries same. Now apply Lemma 15 to get the desired result.

Corollary 17 *An element in $\mathrm{M}_{2^r}(R)$ which commutes with $E(c)(1)^{top}_,$ for $c = e_i$ or e^*_i, $3 \leq i \leq r + 1$, and $E(d)(1)^{tb}_,$ $d = e_2$ or e^*_2, is a scalar matrix.*

Proof: Obvious from the proof of Proposition 16.

An involution on $\mathrm{SUm}_r(R)$, r even

The case when r is even; where the involution can be defined.

Let $\alpha = \prod_{i=1}^{n} S_i$ be a product of Suslin matries $S_i = S_r(v_i, w_i)$, and let α^* denote $\prod_{i=n}^{1} S_i$. If r is even, then $\alpha \mapsto \alpha^*$ is a well defined anti-involution of $\mathrm{SUm}_r(R)$: By Suslin's identities,

$$S_r(v, w) = J_r S_r(v, w)^t J_r^{-1}.$$

Hence, $\alpha^* = J_r \alpha^t J_r^{-1}$, and we are done. From this, it follows that $Z(\mathrm{SUm})_r(R) = \{uI_{2^r} | u^2 = 1\}$, when r is even.

We now discuss the case when r is odd; where we showed that there is an ambiguity to define the involution.

In ([26], Corollary 3.2) we show that if $I_{2^r} = S_r(v_1, w_1) \dots S_r(v_k, w_k)$, for some $\langle v_i, w_i \rangle = 1$, for $1 \leq i \leq k$, then $S_r(v_k, w_k) \dots S_r(v_1, w_1) = uI_{2^r}$, for some unit u with $u^2 = 1$.

Moreover, in ([26], §5) we show that given a unit u with $u^2 = 1$, we can find $S_r(v_i, w_i)$, with $\langle v_i, w_i \rangle = 1$, $1 \leq i \leq k$, for some k, such that

$$\begin{aligned} I_{2^r} &= S_r(v_1, w_1) \dots S_r(v_k, w_k) \\ uI_{2^r} &= S_r(v_k, w_k) \dots S_r(v_1, w_1). \end{aligned}$$

Thus, α^* is defined upto a unit factor when r is odd. This fact is useful to compute $Z(\mathrm{SUm})_r(R)$ when r is odd.

Suslin matrices, Orthogonal transformations

The Fundamental property of Suslin matrices enables one to define an action of the group $\mathrm{SUm}_r(R)$ on the Suslin space. One associates a linear transformation T_g of the Suslin space with a Suslin matrix g, via

$$T_g(x, y) = (x', y'),$$

where $gS_r(x, y)g^* = S_r(x', y')$. Moreover, if g is a product of Suslin matrices $S_r(v_i, w_i)$, with $v_i \cdot w_i^t = 1$, for all i, then $T_g \in \mathrm{O}_{2(r+1)}(R)$, i.e.

$$\langle T_g(v, w), T_g(s, t) \rangle = \langle (v, w), (s, t) \rangle = v \cdot w^t + s \cdot t^t.$$

Translating the Fundamental identities

Theorem 18 ([26], Corollary 4.2) *The above action induces a canonical homomorphism $\varphi : \mathrm{SUm}_r(R) \to \mathrm{SO}_{2(r+1)}(R)$, with*

$$\varphi(S_r(v, w)) \quad = \quad T_{S_r(v,w)} = \tau_{(v,w)} \circ \tau_{(e_1 e_1)},$$

where $\tau_{(v,w)}$ is the standard reflection with respect to the vector $(v,w) \in R^{2(r+1)}$ given by the formula

$$\tau_{(v,w)}(s,t) \quad = \quad \langle v,w\rangle(s,t) - (\langle v,t\rangle + \langle s,w\rangle)(v,w).$$

The matrix of the linear transformation was also calculated in ([23], Chapter 5, Lemma 5.2.1).

Lemma 19 *Let R be a commutative ring with identity. Let $v, w \in \mathrm{Um}_{r+1}(R)$, then the matrix of the linear transformation $T_{S_r(v,w)}$ with respect to the (ordered) basis $\{S_r(e_1,0), S_r(e_2,0), \ldots, S_r(e_{r+1},0), S_r(0,e_1), S_r(0,e_2), \ldots, S_r(0,e_{r+1})\}$ is*

$$\left(\mathrm{I} - (v,w)^t(w,v)\right)\left(\mathrm{I} - (e_1,e_1)^t(e_1,e_1)\right).$$

In particular, for $v = e_1\varepsilon$, $w = e_1{\varepsilon^t}^{-1}$ for some $\varepsilon \in \mathrm{SL}_{r+1}(R)$, the matrix of $T_{S_r(v,w)}$ is the commutator $\left[\varepsilon^t \perp \varepsilon^{-1}, (\mathrm{I} - (e_1,e_1)^t(e_1,e_1))\right]$.

Elementary orthogonal matrices and reflections

Let π denote the permutation $(1\ r+1)\ldots(r\ 2r)$ corresponding to the form $I_r \top I_r$. The **elementary orthogonal matrices** over R is defined as

$$oe_{ij}(z) \quad = \quad I_{2r} + ze_{ij} - ze_{\pi(j)\pi(i)}, \text{ if } i \neq \pi(j) \text{ and } i < j,$$

where $1 \leq i \neq j \leq 2r$, and $z \in R$.

The **elementary orthogonal group** $\mathrm{EO}_{2r}(R)$ is a subgroup of $\mathrm{SO}_{2r}(R)$ generated by the matrices $oe_{ij}(z)$, where $1 \leq i \neq \pi(i) \neq j \leq 2r$, and $z \in R$.

We showed in [24] that every elementary orthogonal transformation can be written as a product of reflections. In fact, the standard generators of $\mathrm{EUm}_r(R)^{tb}$ map onto the standard generators of $\mathrm{EO}_{2(r+1)}(R)$, when r is even. Now apply:

Proposition 20 *Let $\lambda \in R$. For $r \geq 2$, $2 \leq i \neq j \leq r+1$, and $j \neq \pi(i)$, one has, w.r.t. the splitting given in Lemma 13,*

$$\begin{aligned}
oe_{1i}(\lambda) \quad &= \quad \tau_{(e_1-e_j,e_1)} \circ \tau_{(-(1-\lambda)e_1+e_j,-e_1+\lambda e_j)} \circ \tau_{(e_1-e_j,e_1-e_i)} \\
&\quad \circ \tau_{(-(1+\lambda)e_1+e_j,-e_1-\lambda e_j+e_i)} \circ \tau_{(e_1,e_1-e_i)} \circ \tau_{(-e_1,-e_1+\lambda e_j+e_i)} \\
&\quad \circ \tau_{(e_1,e_1-\lambda e_j)} \circ \tau_{(e_1,e_1)} = T_{S_r(e_1,e_1E_{1i}(-\lambda)),^{top}} \\
oe_{i1}(\lambda) \quad &= \quad \tau_{(e_1,e_1-e_j)} \circ \tau_{(-e_1-\lambda e_j,-(1+\lambda)e_1+e_j)} \circ \tau_{(e_1-e_i,e_1-e_j)} \\
&\quad \circ \tau_{(-e_1+\lambda e_j+e_i,(\lambda-1)e_1+e_j)} \circ \tau_{(e_1-e_i,e_1)} \circ \tau_{(-e_1-\lambda e_j+e_i,-e_1)} \\
&\quad \circ \tau_{(e_1+\lambda e_j,e_1)} \circ \tau_{(e_1,e_1)} = T_{S_r(e_1E_{1i}(\lambda),e_1),^{bot}} \\
oe_{\pi(1)i}(\lambda) \quad &= \quad \tau_{(e_1,e_1-\lambda e_j)} \circ \tau_{(-e_1,-e_1+\lambda e_j+e_i)} \circ \tau_{(e_1,e_1-e_i)} \\
&\quad \circ \tau_{(-(1+\lambda)e_1+e_j,-e_1-\lambda e_j+e_i)} \circ \tau_{(e_1-e_j,e_1-e_i)} \\
&\quad \circ \tau_{((\lambda-1)e_1+e_j,-e_1+\lambda e_j)} \circ \tau_{(e_1-e_j,e_1)} \circ \tau_{(e_1,e_1)} = T_{S_r(e_1,e_1E_{1i}(-\lambda)),^{bot}} \\
oe_{i\pi(1)}(\lambda) \quad &= \quad \tau_{(e_1+\lambda e_j,e_1)} \circ \tau_{(-e_1-\lambda e_j+e_i,-e_1)} \circ \tau_{(e_1-e_i,e_1)} \\
&\quad \circ \tau_{(-e_1+\lambda e_j+e_i,(\lambda-1)e_1+e_j)} \circ \tau_{(e_1-e_i,e_1-e_j)} \circ \tau_{(-e_1-\lambda e_j,-(1+\lambda)e_1+e_j)} \\
&\quad \circ \tau_{(e_1,e_1-ej)} \circ \tau_{(e_1,e_1)} = T_{S_r(e_1E_{1i}(\lambda),e_1).^{top}}
\end{aligned}$$

We refer the reader to the Appendix where we show how the mathematical software MuPAD helps in the computation of composition of reflections.

Kernel of $\varphi : \mathrm{SUm}_r(R) \longrightarrow \mathbf{SO}_{2(r+1)}(R)$

We compute the kernel of the map φ, and show that it consists of scalars uI_{2^r}, with $u^2 = 1$. This follows from:

Lemma 21 *Let R be a commutative ring in which 2 is invertible. Let $\alpha \in SUm_r(R)$. Suppose that $\alpha S_r(v,w)\alpha^* = S_r(v,w)$, for all $S_r(v,w) \in \mathrm{EUm}_r(R)$. Then α^* centralizes $\mathrm{EUm}_r(R)$. Consequently, α is a scalar uI_{2^r}, for some unit $u \in R$, and $\alpha \in Z(\mathrm{SUm}_r(R))$.*

Note that in the above statement we have replaced $\mathrm{SUm}_r(R)$ by $\mathrm{EUm}_r(R)$ in ([26], Lemma 4.7). This is possible due to the Corollary 17.

The above lemma is really the key to verifying formulas relating to the action of an element of $\mathrm{SUm}_r(R)$ on a Suslin matrix $S_r(v,w)$.

Computational techniques in $\mathbf{SUm}_r(R)$

We illustrate different computational techniques which help to prove the relations in the group $\mathrm{EUm}_r(R)$, etc. Each method has its own merit. Here we collate **five** such methods.

(1) **Direct Computational Method:** In this method we directly evaluate both sides of the relation using the properties of Suslin matrices as in Lemma 6, and show that they are equal.

(2) **Circle Type Method:** In this method we arrange the matrix block entries in a particular way. The arrangement helps us to do the matrix multiplication easily; as well as gives an inductive framework. We now define this particular type of arrangement in the following definition.

Definition 22 *Let R be a commutative ring with 1. For α, $\beta \in \mathrm{M}_2(\mathrm{M}_n(R))$, $r \geq 1$, say $\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$, $\beta = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}$, where each $\alpha_{ij}, \beta_{ij} \in \mathrm{M}_n(R)$, $1 \leq i,j \leq 2$. We denote by $\alpha \odot \beta$ (read as 'α circles β') the matrix*

$$\begin{pmatrix} \alpha_{11} & 0 & 0 & \alpha_{12} \\ 0 & \beta_{11} & \beta_{12} & 0 \\ 0 & \beta_{21} & \beta_{22} & 0 \\ \alpha_{21} & 0 & 0 & \alpha_{22} \end{pmatrix} \in \mathrm{M}_4(\mathrm{M}_n(R)).$$

Definition 23 *A matrix $\alpha \in \mathrm{M}_{2r}(R)$ is said to be* **circled type** *if there eixsts $\beta, \gamma \in \mathrm{M}_r(R)$ such that $\alpha = \beta \odot \gamma$.*

(Suslin Matrices of Circled Type)
Let R be a commutative ring with 1 and let $v = (a_0, a_1, \cdots, a_r), w = (b_0, b_1, \cdots, b_r) \in \mathrm{M}_{1r+1}(R)$. The Suslin matrix $S_r(v, w)$ is of circled type if and only if $a_1 = 0 = b_1$.

In this case, one observes that

$$S_r(v, w) = S_{r-1}(v_1, w_1) \odot S_{r-1}(v_1^{\odot}, w_1^{\odot}),$$

where $v_1 = (a_0, a_2, \cdots, a_r), w_1 = (b_0, b_2, \cdots, b_r)$, $v_1^{\odot} = (a_0, -b_2, a_3, \cdots, a_r)$, and $w_1^{\odot} = (b_0, -a_2, b_3, \cdots, b_r)$.

(3) **Action Method:** Suppose we expect some relation LHS = RHS. In this method we first show that the action of LHS on $S_r(v, w)$ and the action of RHS on $S_r(v, w)$ are same. Using this equality, one can show via Lemma 20 that LHS$\cdot$RHS$^{-1} \in Z(\mathrm{SUm}_r(R))$ and then by some argument (generic argument), one can show that both LHS and RHS are equal.

(4) **Method of Reflection:** This is a refinement of the previous method. Again see the action on $S_r(v, w)$ as above. However, do show the equality compute via the homomorphism in Theorem 16, $\varphi : \mathrm{SUm}_r(R) \to \mathrm{SO}_{2(r+1)}(R)$ whose image is a composition of two reflections. So instead of multiplying matrices, one plays with pairs of rows (v, w) of unit length. Finally to check equality, use the fact that $\varphi : \mathrm{EUm}_r(R) \to \mathrm{EO}_{2(r+1)}(R)$ is surjective and $\ker \varphi \subset Z(\mathrm{SUm}_r(R))$.

(5) **Orthogonal Matrix Method:** In this method, we evaluate the image of both LHS and RHS under φ in the matrix form and do the computation in $\mathrm{EO}_{2(r+1)}(R)$ and show that both the images are the same. Using the surjectivity of φ, one can come back to $\mathrm{EUm}_r(R)$ and using some argument as in the previous method one can say that both sides are equal.

Quillen–Suslin theory for $\mathbf{EUm}_r(R[X])$

The image of φ contains all even products of reflections, and hence, in particular, all elementary orthogonal matrices.

Thus, all questions concerning the group $\mathrm{EUm}_r(R)$ can be reduced to the corresponding questions regarding elementary orthogonal matrices. For example, one has a Quillen–Suslin theory for the elementary orthogonal groups $\mathrm{EO}_{2n}(R[X])$ due to results of Suslin–Kopeiko in [62] - both the Local Global Principle and the Monic Inversion Principle of Quillen–Suslin hold for the Elementary Unimodular vector group $\mathrm{EUm}_r(R[X])$. From the Local Global Principle, or otherwise, one can conclude that $\mathrm{EUm}_r(R[X])$ is a normal subgroup of $\mathrm{SUm}_r(R[X])$, for $r > 1$.

$\mathbf{SUm}_r(R)/\mathbf{EUm}_r(R) \hookrightarrow \mathbf{SO}_{2(r+1)}(R)/\ \mathbf{EO}_{2(r+1)}(R)$

In this subsection, we recall the main work of Jose–Rao in [25] where they show how the Fundamental property led to showing that the quotient of the Special Unimodular vector group by its Elementary unimodular vector group sits inside the orthogonal quotient; *viz.* it was shown in ([26], Theorem 4.14) that the induced map φ on the quotients is an injection, whence $\mathrm{SUm}_r(R)/\mathrm{EUm}_r(R)$ is a subgroup of the orthogonal quotient $\mathrm{SO}_{2(r+1)}(R)/\mathrm{EO}_{2(r+1)}(R)$

This is clear from Proposition 20 which shows that φ maps $\mathrm{EUm}_r(R) \to \mathrm{EO}_{2(r+1)}(R)$ given by $\varphi(S_r(v,w)) = T_{S_r(v,w)}$ is surjective. Moreover, one has the kernel of the map $\varphi : \mathrm{SUm}_r(R) \to \mathrm{SO}_{2(r+1)}(R)$ is contained in $\mathrm{Z}(\mathrm{SUm}_r(R))$.

R. Hazrat and N. Vavilov, using ideas of A. Bak in [3], have shown in [21] that the orthogonal quotient group is nilpotent. Hence, the unimodular vector group quotient $\mathrm{SUm}_r(R)/\mathrm{EUm}_r(R)$ is a nilpotent group, for $r > 1$.

Injective stability for the $\mathbf{K}_1$ orthogonal functor

We used results in ([55], §4) in ([54], Corollary 2.7) to show that the injective stability for the orthogonal $\mathrm{K}_1\mathrm{O}$ functor cannot fall, in general for an affine algebra. We recapitulate that result here. Thus the Suslin matrices have been found useful in the context of injective stability bounds of the orthogonal $\mathrm{K}_1\mathrm{O}$ functors.

Before that we recall yet another lemma from [26].

Lemma 24 *Let $S_r(v,w)$, $S_r(v',w')$, $r > 1$, $\langle v,w\rangle = \langle v',w'\rangle = 1$, be Suslin matrices. If $S_r(v,w) \in S_r(v',w')\mathrm{EUm}_r(R)$, then*

(i) if r is even $\chi_2(v) \overset{\mathrm{E}}{\sim} \chi_2(v')$,

(ii) if r is odd $\chi_4(v) \overset{\mathrm{E}}{\sim} \chi_4(v')$.

Lemma 25 *Let A be a an affine algebra of dimension d over a perfect field k, of characteristic $\neq 2$, and with c.d.$_2(k) \leq 1$. Assume that $mA = A$ for some $m > 0$. If $v \in \mathrm{Um}_{d+1}(A)$ then there is a row of the form $(v_1^m, \ldots, v_{d+1})$ in the elementary orbit of v.*

Theorem 26 ([54], Theorem 2)

Let A be a an affine algebra of dimension d over an algebraically clsoed field, or a non-singular one over a perfect C_1-field. Assume $2A = A$. If the natural map

$$\rho_{\mathrm{O}} : \frac{\mathrm{SO}_{2(d+1)}(A)}{\mathrm{EO}_{2(d+1)}(A)} \leftrightarrow \frac{\mathrm{SO}_{2(d+2)}(A)}{\mathrm{EO}_{2(d+2)}(A)}$$

is an isomorphism, then every unimodular $(d+1)$-row over A can be completed to an elementary matrix. However, $\mathrm{Um}_{d+1}(A) = e_1\mathrm{E}_{d+1}(A)$ does not hold in general.

Proof: Let d be odd. Let $v \in \mathrm{Um}_{d+1}(A)$. Choose any w with $v \cdot w^t = 1$. By Lemma 19 the matrix of the linear transformation $T_{S_d(v,w)}$ is a commutator, hence stably elementary orthogonal. The hypothesis enables us to conclude that it is elementary orthogonal. By Lemma 24, $S_d(v,w) \in \mathrm{EUm}_d(A)$. Moreover, by Lemma 24, $\chi_4(v) = 1$.

By Lemma 25 as $2A = A$, every row $v \in \mathrm{Um}_{d+1}(A)$ is a $\chi_4(v')$, for some $v' \in \mathrm{Um}_{d+1}(A)$. The result follows.

A similar argument can be given when d is even. Using the corresponding results of [26].

Corollary 27 *There exist affine algebras A of dimension $d \geq 2$ over a perfect C_1-field k for which the injective stability estimate for $\mathrm{K}_1\mathrm{O}(A)$ is not less than $2(d+2)$.*

Theorem 28 *Let A be a local ring of dimension d, with $2A = A$. If the natural map $\mathrm{SO}_{2(d+1)}(A[X])/\mathrm{EO}_{2(d+1)}(A[X]) \longrightarrow \mathrm{K}_1\mathrm{O}(A)$ is an isomorphism, then every unimodular $(d+1)$-row over $A[X]$ can be completed to an elementary matrix.*

Corollary 29 *There exists an affine algebras A of dimension 3, and a maximal ideal $\mathfrak{m}$ of A, for which the injective stability estimate for $\mathrm{K}_1\mathrm{O}(A_{\mathfrak{m}}[X])$ is not 8.*

Proof: In ([55], §4), it is shown that if $A = k[X,Y,Z]/(Z^7 - X^2 - Y^3)$, where $k = \mathbb{C}$ or a sufficiently large field, then $\mathrm{Um}_3(A[T,T^{-1}][X],(X)) \neq e_1\mathrm{E}_3(A[T,T^{-1}][X])$. Note that A is regular except at the maximal ideal $\mathfrak{m} = (X,Y,Z)$. Hence, by Suslin's version of the Local Global Principle in [61], and T. Vorst's theorem in [77], it follows that there is a maximal ideal $\mathfrak{M}$ containing $\mathfrak{m}[T,T^{-1}]$ such that $e_1\mathrm{E}_3(A[T,T^{-1}]_{\mathfrak{M}}[X]) \neq \mathrm{Um}_3(A[T,T^{-1}]_{\mathfrak{M}}[X])$. Now apply Theorem 28.

Appendix: Reflections via MuPAD

We define the reflection function $\tau_{(x,y)}(z,w)$ via MuPAD for $r=4$, where x,y,z,w are vectors of length 5 as follows: In all the commands given below, we suppress the output by putting colon (:) at the end of each input statement.

To define the function $\tau_{(x,y)}(z,w)$, we need to define the vectors x,y,z,w. The vectors `x, y, z, w` are defined as:

- `x := matrix([[x0,x1,x2,x3,x4]]):`
- `y := matrix([[y0,y1,y2,y3,y4]]):`
- `z := matrix([[z0,z1,z2,z3,z4]]):`
- `w := matrix([[b0,b1,b2,b3,b4]]):`
- `assume(Type::Real):`
- `f:=(x,y,z,w) -> linalg::scalarProduct(x,y) * matrix([z,w])`
  `-(linalg::scalarProduct(x,w) + linalg::scalarProduct(y,z)) * matrix([x,y]):`

The above statement defines the function

$$f(x,y,z,w) = \langle x,y\rangle(z,w) - (\langle x,w\rangle + \langle y,z\rangle)(x,y).$$

Thus $f(x,y,z,w)$ will give the value of $\tau_{(x,y)}(z,w)$.

As an illustration, we give the computation we did in the proof of Proposition 20 for $i=5, j=3$. The computation uses the following vectors:

- `v := matrix([[a0,a1,a2,a3,a4]]):`
- `w := matrix([[b0,b1,b2,b3,b4]]):`
- `e1 := matrix([[1,0,0,0,0]]):`
- `ei := matrix([[0,0,0,0,1]]):`
- `ej := matrix([[0,0,1,0,0]]):`

In the following input statements, we use `L` for λ. We first evaluate $\tau_{(e_1-e_j,e_1)} \circ \tau_{(-(1-\lambda)e_1+e_j,-e_1+\lambda e_j)} \circ \tau_{(e_1-e_j,e_1-e_i)} \circ \tau_{(-(1+\lambda)e_1+e_j,-e_1-\lambda e_j+e_i)} \circ \tau_{(e_1,e_1-e_i)} \circ \tau_{(-e_1,-e_1+\lambda e_j+e_i)} \circ \tau_{(e_1,e_1-\lambda e_j)} \circ \tau_{(e_1,e_1)}$ at (v,w).

- `AA := simplify(f(e1,e1,v,w))`

  Output:

$$\begin{aligned} v_1 &= (-b_0,a_1,a_2,a_3,a_4) \text{ and} \\ w_1 &= (-a_0,b_1,b_2,b_3,b_4) \end{aligned}$$

- `AB := simplify(f(e1,e1-L*ej,AA[1],AA[2]))`

  Output:

$$\begin{aligned} v_2 &= (a_0+LPa_2,a_1,a_2,a_3,a_4) \text{ and} \\ w_2 &= (b_0+LPa_2,b_1,b_2-L^2Pa_2-LPa_0-LPb_0,b_3,b_4) \end{aligned}$$

- `AC := simplify(f(-e1,-e1+L*ej+ei,AB[1],AB[2]))`

  Output:

$$\begin{aligned} v_3 &= (a_4 - b_0, a_1, a_2, a_3, a_4) \text{ and} \\ w_3 &= (-a_0 + a_4, b_1, b_2 - LPa_4, b_3, a_0 - a_4 + b_0 + b_4 + LPa_2) \end{aligned}$$

- `AD := simplify(f(e1,e1-ei,AC[1],AC[2]))`

  Output:

$$\begin{aligned} v_4 &= (a_0, a_1, a_2, a_3, a_4) \text{ and} \\ w_4 &= (b_0, b_1, b_2 - LPa_4, b_3, b_4 + LPa_2) \end{aligned}$$

- `AE := simplify(f(-(1+L)*e1+ej,-e1-L*ej+ei,AD[1],AD[2]))`

  Output:

$$\begin{aligned} v_5 &= (a_4 - b_0 + b_2 - L^2Pa_2 - L^2Pa_4 - L^2Pb_0 - LPa_0 - LPa_2 - 2PLPb_0 + LPb_2, \\ &\quad a_1, a_0 + a_2 - a_4 + b_0 - b_2 + L.a_2 + L.a_4 + L.b_0, a_3, a_4) \text{ and} \\ w_5 &= (-a_0 + a_4 + b_2 - LPa_2 - LPa_4 - LPb_0, b_1, \\ &\quad b_2 - L^2Pa_2 - L^2Pa_4 - L^2Pb_0 - LPa_0 - LPb_0 + LPb_2, b3, \\ &\quad a_0 - a_4 + b_0 - b_2 + b_4 + 2.L.a_2 + L.a_4 + L.b_0) \end{aligned}$$

- `AF := simplify(f(e1-ej,e1-ei,AE[1],AE[2]))`

  Output:

$$\begin{aligned} v_6 &= (a_0 - L^2Pa_2 - L^2Pa_4 - L^2Pb_0 - LPa_0 + LPa_2 + LPa_4 + LPb_2, a_1, \\ &\quad a_2 - LPa_2 - LPb_0, a_3, a_4) \text{ and} \\ w_6 &= (b_0 + LPa_2 + LPb_0, b_1, \\ &\quad b_2 - L^2Pa_2 - L^2Pa_4 - L^2Pb_0 - LPa_0 - LPb_0 + LPb_2, b_3, b_4 - LPb_0) \end{aligned}$$

- `AG := simplify(f(-(1-L)*e1+ej,-e1+L*ej,AF[1],AF[2]))`

  Output:

$$\begin{aligned} v_7 &= (-b_0 + b_2, a_1, a_0 + a_2 + b_0 - b_2 + L.a_4, a_3, a_4) \text{ and} \\ w_7 &= (-a_0 + b_2 - L.a_4, b_1, b_2, b_3, b_4 - L.b_0) \end{aligned}$$

- `AH := simplify(f(e1-ej,e1,AG[1],AG[2]))`

  Output:

$$\begin{aligned} v_8 &= (a_0 + LPa_4, a_1, a_2, a_3, a_4) \text{ and} \\ w_8 &= (b_0, b_1, b_2, b_3, b_4 - LPb_0) \end{aligned}$$

Note that this value is same as $oe_{15}(\lambda)\begin{pmatrix} v^t \\ w^t \end{pmatrix}$, where

$$oe_{15}(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\lambda & 0 & 0 & 0 & 1 \end{pmatrix}.$$

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