IWAHORI – HECKE ALGEBRAS

I GOAL EXAMINE REPRESENTATIONS OF p-ADIC REDUCTIVE GPS USING HECKE ALGEBRAS

MORE PRECISELY : LET $G = GL_n(\mathbb{Q}_p)$, REGARDED AS A TOP GP W/ ITS p-ADIC TOPOLOGY (MOST STATEMENTS WORK FOR A SPLIT CONNECTED REDUCTIVE GROUP DEFINED OVER $\mathbb{Q}_p$)

FIX ALSO AN ALG CLOSED FIELD C (OF ARB. CHAR FOR NOW, FOR COEFFS)

WE ARE INTERESTED IN THE CATEGORY

CTS

$$\mathrm{REP}_C(G) := \left\{ \begin{matrix} \text{SMOOTH REPS } V \\ \text{OF } G \text{ OVER } C \end{matrix} \right\}$$

DISC TOP

$$\left(\text{SMOOTH} = \forall v \in V, \ \mathrm{STAB}_G(v) \text{ IS OPEN IN } G \right)$$

WHY ARE WE INTERESTED IN THIS CATEGORY ?

0) REP THEORY SEEKS TO CLASSIFY REPS OF GPS

1) LLC THE SIMPLE OBJECTS IN $\mathrm{REP}_C(G)$ ARE (CONJECTURALLY) PARAMETRIZED BY GALOIS REPRESENTATIONS

$$\mathrm{GAL}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \xrightarrow{\text{CTS}} GL_n(C)$$

(ROUGHLY)

$$= \hat{G}(C)$$

LANGLANDS DUAL GP

$\rightsquigarrow$ KNOWLEDGE OF $\mathrm{Rep}_{C}(G)$ HAS CONSEQUENCES FOR UNDERSTANDING HIGHER RAMIFICATION IN $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, STRUCTURE OF $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (CHENEVIER - CLOZEL), GLOBAL ARITHMETIC QUESTIONS (FLT), ETC

2) REPS OF G ARE VERY HANDS-ON (IN CERTAIN CASES), AND ARE THEREFORE A GOOD STARTING PT

// **TODAY** $C = \mathbb{C}$

HOW DO WE STUDY $\mathrm{Rep}_{\mathbb{C}}(G)$?

WE CAN ARRANGE REPS BY "DEPTH" OR "LEVEL"

"LEVEL 0" LET $K = GL_n(\mathbb{Z}_p)$

$\rightsquigarrow$ K IS A MAXIMAL COMPACT OPEN SUBGP OF G

CONSIDER THE FOLLOWING SUBSET OF $\mathrm{Rep}_{\mathbb{C}}(G)$:

$$\left\{ \begin{matrix} \text{SMOOTH IRREPS } V \\ \text{OF } G \text{ S.T.} \\ V^K \neq 0 \end{matrix} \right\} / \cong$$

$(V^K = \{ v \in V : k \cdot v = v \ \forall k \in K \}$ "K-FIXED VECTORS")

EX THIS SET CONTAINS

- TRIVIAL REP:

$$V = \mathbb{C} \quad \text{w/} \quad g \cdot v = v \quad \forall g \in G,\ v \in V$$

- "UNRAMIFIED PRINCIPAL SERIES":

$(G = GL_2(\mathbb{Q}_p))$ $\quad V = \mathcal{C}^{\text{loc cst}}(\mathbb{P}^1(\mathbb{Q}_p), \mathbb{C})$ w/

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f\right)([X:Y]) = f([dX - bY : -cX + aY])$$

$$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Q}_p),\quad f \in V$$

Q HOW DO WE UNDERSTAND REPS V w/ $V^K \neq 0$?

FACT 1 V^K HAS A "RESIDUAL ACTION" OF

$$\mathcal{H}(G,K) = \left\{ f : G \to \mathbb{C} : \begin{array}{l} \cdot\ f(kgk') = f(g)\ \forall g \in G,\ k,k' \in K \\ \cdot\ f \text{ HAS CPCT SUPP} \end{array} \right\}$$

"SPHERICAL HECKE ALGEBRA"

+ CONVOLUTION PRODUCT (WILL DESCRIBE THIS EXPLICITLY LATER)

FACT 2 CAN DESCRIBE ALGEBRA STRUCTURE OF $\mathcal{H}(G,K)$ EXPLICITLY: SATAKE ISOMORPHISM GIVES

WEYL GP OF T

$$\mathcal{H}(G,K) \xrightarrow[\sim]{S} \mathcal{H}(T, T \cap K)^W \cong \mathbb{C}[T/T\cap K]^W$$

C-ALG HOM

$$\mathbb{1}_{K(p,1)K} \longmapsto p^{1/2}\left(\mathbb{1}_{(T\cap K)(p,1)} + \mathbb{1}_{(T\cap K)(1,p)}\right) \qquad \cong \mathbb{C}[X_*(T)]^W$$

$$\mathbb{1}_{K(p,p)} \longmapsto \mathbb{1}_{(T\cap K)(p,p)}$$

MAX'L TORUS OF G E.G $T = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}$

COCHARACTER GP OF T

$= \mathrm{Hom}(\mathbb{G}_m, T)$

IN PART'R, $\mathcal{H}(G, K)$ IS COMM.

FACT 3   IF V IRRED + $V^K \neq 0$, THEN $\dim(V^K) = 1$

SO

$$\text{IRRED } V \text{ w/ } V^K \neq 0 \quad \overset{①}{\longleftrightarrow} \quad V^K \text{ w/ } \mathcal{H}(G,K)\text{-ACTION}$$

$$\overset{③}{\longleftrightarrow} \quad \mathbb{C}\text{-ALG HOM } \mathcal{H}(G,K) \longrightarrow \mathbb{C}$$

$$\overset{②}{\longleftrightarrow} \quad \mathbb{C}\text{-ALG HOM } \mathbb{C}[X_*(T)]^W \longrightarrow \mathbb{C}$$

DUAL TORUS $\subset \hat{G}$

$$\hat{T}(\mathbb{C}) = X_*(\hat{T}^\vee) \otimes_{\mathbb{Z}} \mathbb{C}^*$$
$$\lambda(\alpha) \longleftarrow\!\shortmid \lambda \otimes \alpha$$
$$= X^*(T^\vee) \otimes_{\mathbb{Z}} \mathbb{C}^*$$
$$= \mathrm{Hom}_{\mathbb{Z}}(X_*(T), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}^*$$
$$= \mathrm{Hom}_{\mathbb{Z}}(X_*(T), \mathbb{C}^*)$$
$$= \mathrm{Hom}_{\mathbb{C}\text{-Alg}}(\mathbb{C}[X_*(T)], \mathbb{C})$$

$$\longleftrightarrow \quad \text{ELT } \hat{t} \text{ OF } \hat{T}(\mathbb{C})/W$$

$$\longleftrightarrow \quad r : W_p \twoheadrightarrow \langle \text{FROB} \rangle \longrightarrow \hat{G}(\mathbb{C})$$
$$\text{FROB} \longmapsto \hat{t}$$

W-D UP TO $\hat{G}$- CONJ

THIS GIVES AN INSTANCE OF THE UNRAMIFIED LLC

"LEVEL 1"   NOW LET I = IWAHORI SUBGP OF G

= PREIMAGE UNDER

$GL_n(\mathbb{Z}_p) \twoheadrightarrow GL_n(\mathbb{F}_p)$

OF BOREL SUBGP

$$= \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}$$

$$\equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod p$$

CONSIDER

$$\left\{ \begin{array}{c} \text{SMOOTH IRREPS } V \\ \text{OF } G \text{ S.T.} \\ V^I \neq 0 \end{array} \right\} / \cong$$

SOMETIMES CALLED "IWAHORI - SPHERICAL"

CONTAINS PREVIOUS REPS + MORE :

EX $\quad St := \mathcal{C}^{\text{LOC CST}}(\mathbb{P}^1(\mathbb{Q}_p), \mathbb{C}) / \{\text{CST FNS}\} \circlearrowleft GL_2(\mathbb{Q}_p)$

IS CONTAINED ABOVE, BUT NOT IN K - SPHERICAL REPS

IE, $St^I \neq 0$, $St^K = 0$

FACT AS BEFORE, V^I OBTAINS A "RESIDUAL ACTION" OF

$$\mathcal{H}(G, I) = \left\{ f : G \longrightarrow \mathbb{C} \;:\; \begin{array}{l} \cdot\ f(igi') = f(g) \quad \forall g \in G,\ i, i' \in I \\ \cdot\ f \text{ COMPACT SUPP} \end{array} \right\}$$

+ CONVOLUTION PRODUCT (MORE LATER)

"IWAHORI - HECKE ALGEBRA"

N.B. $\mathcal{H}(G, I)$ IS NO LONGER COMMUTATIVE, BUT WE CAN STILL EXPLICITLY DESCRIBE ITS STRUCTURE (NEXT TIME)

MOREOVER, IF V IRRED AND $V^I \neq 0$, THEN

V^I IS A SIMPLE $\mathcal{H}(G, I)$-MOD

USING IDEAS FROM GEOMETRIC REP THEORY, (K-HOMOLOGY OF STEINBERG VARIETY OF TRIPLES) KAZHDAN-LUSZTIG CLASSIFIED SIMPLE $\mathcal{H}(G, I)$-MODS AND SHOWED THEY ARE NATURALLY IN BIJECTION W/ CERTAIN WEIL-DELIGNE REPS (ONLY FOR GPS W/ CONNECTED CENTER; W-D REPS WHICH ARE UNRAMIFIED ON W_p, BUT N CAN BE ANYTHING COMPATIBLE W/ $r(\text{FROB})$)

IN BOTH CASES (K AND I), PASSING FROM IRREPS TO SIMPLE MODULES OVER $\mathcal{H}(G, K)$ OR $\mathcal{H}(G, I)$ ALLOWS US TO EXPLICITLY CONSTRUCT INSTANCES OF LLC (FOR "ALL" G AT ONCE)

EVEN BETTER: FOR THE IWAHORI, THE BIJECTION B/W IRREPS V W/ $V^I \neq 0$ AND SIMPLE $\mathcal{H}(G, I)$-MODS IS INDUCED FROM AN EQUIVALENCE OF CATS:

THM (BOREL, BERNSTEIN)

EQUIV OF CATS

$$\mathrm{Rep}^I_{\mathbb{C}}(G) := \{ V \in \mathrm{Rep}_{\mathbb{C}}(G) : \langle G \cdot V^I \rangle = V \} \xrightarrow{\sim} H(G,I)\text{-MOD}$$

$$V \longmapsto V^I$$

BLOCK OF $\mathrm{REP}_{\mathbb{C}}(G)$

EQUIV MATCHES SIMPLE OBJECTS ON BOTH SIDES

- GIVES INFO ABOUT THE CATEGORY $\mathrm{REP}_{\mathbb{C}}(G)$
- NOT TRUE FOR K

"HIGHER LEVEL": LEADS TO THEORY OF TYPES OF BUSHNELL – KUTZKO

II LAST TIME: MOTIVATION

TODAY: STRUCTURE OF $\mathcal{H}(G, I)$

SLIGHTLY DIFF'T THAN LAST TIME

NOTATION

$G = SL_n(\mathbb{Q}_p)$ (BUT WORKS FOR ANY SEMISIMPLE SIMPLY CONNECTED GP $/\mathbb{Q}_p$)

$T =$ MAX'L TORUS $\begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}$

$B = TU =$ BOREL SUBGP $\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$

$I =$ IWAHORI SUBGP $\begin{pmatrix} \mathbb{Z}_p^\times & & \mathbb{Z}_p \\ & \ddots & \\ p\mathbb{Z}_p & & \mathbb{Z}_p^\times \end{pmatrix}$

$N = N_G(T) =$ NORMALIZER OF T

$W = N/T =$ WEYL GP

$\cong S_n$ PERMUTATION MATRICES

$q := p$

GROUP-THEORETIC FACTS (BRUHAT-TITS, IWAHORI-MATSUMOTO)

THE PAIR (I, N) IS A BN PAIR. MORE PRECISELY:

- $\tilde{W} := N/T\cap I$ IS A COXETER GROUP, WITH GENS $\tilde{S}$ + LENGTH FN ℓ "AFFINE WEYL GROUP"
- THE S.E.S.

$$1 \longrightarrow T/T\cap I \longrightarrow N/T\cap I \longrightarrow N/T \longrightarrow 1$$

$$\begin{pmatrix} \text{\rotatebox{90}{$\cong$}} \\ \mathbb{Z}^{n-1} \end{pmatrix} \qquad \| \; \tilde{W} \qquad \| \; W$$

SPLITS, GIVING $\tilde{W} \cong W \ltimes (T/T\cap I)$ NATURAL ACTION

- (BRUHAT DECOMPOSITION)

$$G = \coprod_{w \in \widetilde{W}} I \dot{w} I$$

- IF $w, w' \in \widetilde{W}$ SATISFY $\ell(ww') = \ell(w) + \ell(w')$, THEN

$$I \dot{w} I \dot{w}' I = I \dot{w} \dot{w}' I$$

- IF $s \in \widetilde{S}$, THEN

$$I \dot{s} I \dot{s} I = I \dot{s} I \sqcup I$$

- $[I \dot{w} I : I] := | I \dot{w} I / I | = q^{\ell(w)}$

EX

- $W \cong S_n$ w/ GENS

$$s_1 = \begin{pmatrix} \boxed{\begin{matrix} & 1 \\ -1 & \end{matrix}} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 & & & \\ & \boxed{\begin{matrix} & 1 \\ -1 & \end{matrix}} & & \\ & & 1 & \\ & & & \ddots \\ & & & & 1 \end{pmatrix}, \ldots, s_{n-1} = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \boxed{\begin{matrix} & 1 \\ -1 & \end{matrix}} \end{pmatrix}$$

- $\widetilde{W} = W \ltimes (T / T \cap I)$

$= $ "AFFINE WEYL GP OF TYPE $\widetilde{A}_{n-1}$"

W/ GENS $\underbrace{s_1, \ldots, s_{n-1}, s_0}_{\tilde{S}} = \begin{pmatrix} 1 & & -p^{-1} \\ & \ddots & \\ p & & 1 \end{pmatrix}$

NOTE THAT $s_i^2 = 1$ IN $\tilde{W}$

RECALL

$$\mathcal{H} := \mathcal{H}(G, I) = \left\{ f : G \longrightarrow \mathbb{C} : \begin{array}{l} \bullet\ f \text{ HAS CPCT SUPP} \\ \bullet\ f(igi') = f(g) \\ \quad \forall\, i, i' \in I,\ g \in G \end{array} \right\}$$

$\Longrightarrow$ BY BRUHAT DECOMP, $\mathcal{H}$ HAS A BASIS

$$\left\{ T_w := \mathbb{1}_{I\dot{w}I} \right\}_{w \in \tilde{W}}$$

THE PRODUCT IS GIVEN BY CONVOLUTION: FOR $f, f' \in \mathcal{H}$,

$$(f * f')(g) = \int_G f(h)\, f'(h^{-1}g)\, d\mu(h)$$

$\mu(I) = 1$

$$= \sum_{h \in G/I} f(h)\, f'(h^{-1}g)$$

BY I-BI-INVARIANCE

FINITE SUM BY CPCT SUPP

N.B. THE PRODUCT MAKES SENSE FOR ANY COEFF RING R (w/ FNS $f: G \to R$)

E.G. $R = \overline{\mathbb{F}}_p$

WHAT IS THE PRODUCT ON THE BASIS T_w?

LEMMA (IWAHORI–MATSUMOTO)

a) IF $w, w' \in \widetilde{W}$ SATISFY $\ell(ww') = \ell(w) + \ell(w')$, THEN

$$T_w T_{w'} = T_{ww'}$$

"BRAID REL'N"

b) IF $s \in \widetilde{S}$, THEN

$$T_s^2 = (q-1) T_s + q T_1$$

"QUADRATIC REL'N"

IDENTITY ELT OF $\mathcal{H}$

EX $G = SL_2(\mathbb{Q}_p)$

$\widetilde{W}$ IS GEN'D BY $s_1 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ AND $s_0 = \begin{pmatrix} & -p^{-1} \\ p & \end{pmatrix}$

IE, ANY $w \in \widetilde{W}$ CAN BE WRITTEN

$$w = \underbrace{s_1 s_0 s_1 s_0 \cdots}\quad \left(\text{OR} \quad \underbrace{s_0 s_1 s_0 s_1 \cdots}\right)$$

"MINIMAL LENGTH EXPRESSION"

THEN

$$T_w = T_{s_1} T_{s_0} T_{s_1} T_{s_0} \cdots \quad (\text{OR } T_{s_0} T_{s_1} T_{s_0} T_{s_1} \cdots)$$

$\Rightarrow$ $\mathcal{H}$ IS GENERATED AS A $\mathbb{C}$-ALGEBRA BY T_{s_1} AND T_{s_0}

RELATIONS :

$$\left.\begin{aligned} T_{s_1}^2 &= (q-1)\,T_{s_1} + q\,T_1 \\ T_{s_0}^2 &= (q-1)\,T_{s_0} + q\,T_1 \end{aligned}\right\}$$

THIS GIVES A PRESENTATION OF $\mathcal{H}$

NOTE IF G IS BIGGER, HAVE MORE RELATIONS AMONG THE T_{s_i} (COMING FROM RELS IN $\widetilde{W}$)

E.G. FOR $SL_3(\mathbb{Q}_p)$, WE HAVE

$s_1 s_2 s_1 = s_2 s_1 s_2$ IN $\widetilde{W}$ $\rightsquigarrow$ $T_{s_1} T_{s_2} T_{s_1} = T_{s_1 s_2 s_1}$

"MINIMAL LENGTH EXPRESSION"

$$\shortparallel$$

$T_{s_2} T_{s_1} T_{s_2} = T_{s_2 s_1 s_2}$

EX $G = GL_2(\mathbb{Q}_p)$

IN THIS CASE, $\widetilde{W} = N/T\cap I$ IS ALMOST A COXETER GROUP : GENERATED BY s_1, s_0 AS ABOVE, AND $\omega = \begin{pmatrix} & 1 \\ p & \end{pmatrix}$

"LENGTH 0", NORMALIZES I

THESE SATISFY $\omega s_1 \omega^{-1} = s_0$

$\omega s_0 \omega^{-1} = s_1$

AND ANY $w \in \widetilde{W}$ CAN BE WRITTEN

$w = \omega^n s_0 s_1 \cdots$ OR $\omega^n s_1 s_0 \cdots$

$\Rightarrow$ $\mathcal{H}$ IS GENERATED BY

T_{s_1} , T_{s_0} AND T_ω

W/ QUAD'C RELS SAME AS BEFORE, BUT

ALSO HAVE

$$T_\omega T_{s_1} T_\omega^{-1} = T_{s_0} \qquad T_\omega T_{s_0} T_\omega^{-1} = T_{s_1}$$

$$T_\omega^2 = T_{\omega^2} \text{ IS CENTRAL}$$

$\rightsquigarrow$ $\mathcal{H}$ IS GENERATED BY T_{s_1} AND T_ω

<u>SKETCH</u> (PF FOR $SL_2(\mathbb{F}_q)$ IN EXERCISE SESSION)

$$(T_w T_{w'})(g) = \sum_{h \in G/I} T_w(h)\, T_{w'}(h^{-1}g)$$

LHS $\neq 0$ $\Rightarrow$ SOME SUMMAND $\neq 0$

$\Rightarrow$ $h \in IwI$, $h^{-1}g \in Iw'I$

$\Rightarrow$ $g \in hIw'I \subset IwIw'I$

- IF $\ell(ww') = \ell(w) + \ell(w')$, $g \in Iww'I$

- IF $s \in \tilde{S}$, $g \in IsI \sqcup I$

SINCE $T_w T_{w'}$ IS I-BI-INV'T, SUFFICES TO TAKE $g' = ww'$, s, 1

E.G.

FOR $G = SL_2(\mathbb{Q}_p)$, $s = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$, $I = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}$,

WE HAVE

$$IsI/I = \bigsqcup_{x \in \mathbb{Z}_p/p\mathbb{Z}_p} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} s \longleftarrow \text{SIZE } q = p$$

SO

$$(T_s^2)(1) = \sum_{h \in G/I} T_s(h) T_s(h^{-1})$$

$$= \sum_{h \in IsI/I} T_s(h^{-1})$$

$$= q$$

$$(T_s^2)(s) = \sum_{h \in IsI/I} T_s(h^{-1}s)$$

$$= \sum_{x \in \mathbb{Z}/p} T_s\left(s^{-1} \begin{pmatrix} 1 & -x \\ & 1 \end{pmatrix} s\right)$$

$$= \cancel{T_s(1)} + \sum_{x \in (\mathbb{Z}/p)^\times} T_s\left(\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}\right)$$

$$= \sum_{x \neq 0} T_s\left(\underbrace{\begin{pmatrix} 1 & x^{-1} \\ & 1 \end{pmatrix} \begin{pmatrix} x^{-1} & \\ & -x \end{pmatrix}}_{\in I} s \underbrace{\begin{pmatrix} 1 & x^{-1} \\ & 1 \end{pmatrix}}_{\in I}\right)$$

$$= \sum_{x \neq 0} T_s(s)$$

$$= q - 1$$

$$\Longrightarrow \quad T_s^2 = (q-1) T_s + q T_1$$

QUAD'C RELS FOR GENERAL SPLIT GP FOLLOWS FROM USING PRINCIPAL SL_2's

CAN PROVE BRAID RELS SIMILARLY □

SUPPOSE NOW THAT V IS A SMOOTH G-REP

THEN $\mathcal{H}$ ACTS ON V^I: IF $v \in V^I$ AND $T_w \in \mathcal{H}$, THEN

$$\boxed{v \cdot T_w = \sum_{h \in I / I \cap \dot{w}^{-1} I \dot{w}} h \dot{w}^{-1} \cdot v \quad \in V^I}$$

EX LET $G = SL_2(\mathbb{Q}_p)$, $V = \mathrm{Ind}_B^G(1_B) = \{ f : G \to \mathbb{C} :$

TRIVIAL

$\cdot\, f$ LOC CST
$\cdot\, f(bg) = f(g) \quad \forall b \in B, g \in G \}$

THEN $V^I = \mathrm{span}\{ f_{BI}, f_{BsI} \}$ $\quad$ $(\mathrm{supp}(f_{BwI}) = BwI)$

AND USING ABOVE FORMULA, CAN CALCULATE (IN EXERCISE SESSION)

$$f_{BI} \cdot T_{s_1} = f_{BsI} \qquad f_{BsI} \cdot T_{s_1} = q f_{BI} + (q-1) f_{BsI}$$

$$f_{BI} \cdot T_{s_0} = (q-1) f_{BI} + q f_{BsI} \qquad f_{BsI} \cdot T_{s_0} = f_{BI}$$

IN PARTICULAR, WE HAVE A S.E.S.

$$0 \longrightarrow 1_G \longrightarrow \mathrm{ind}_B^G(1_B) \longrightarrow ST \longrightarrow 0$$

WHICH GIVES

$$0 \longrightarrow 1_G^I \longrightarrow \mathrm{ind}_B^G(1_B)^I \longrightarrow ST^I \longrightarrow 0$$

1_G^I IS 1-DIM'L, SPANNED BY $f_{BI} + f_{BsI}$, W/ SCALAR ACTION OF $\mathcal{H}$

$$T_{s_1} \longmapsto q \quad , \quad T_{s_0} \longmapsto q$$

ST^I IS 1-DIM'L, SPANNED BY $\overline{f_{BI}}$, W/ SCALAR ACTION OF $\mathcal{H}$

$$T_{s_1} \longmapsto -1 \quad , \quad T_{s_0} \longmapsto -1$$

RMK WHERE DOES PRODUCT / ACTION FORMULA COME FROM?

FROBENIUS RECIPROCITY: IF U IS A REP OF I AND V IS A REP OF G, THEN

$$\mathrm{Hom}_I(U, V|_I) \cong \mathrm{Hom}_G(\text{c-ind}_I^G(U), V)$$

$$f \longmapsto F$$

$$F([\![g, v]\!]) = g \cdot f(v)$$

AND

$$\mathrm{Hom}_I(1_I, V|_I) = V^I$$

$$f \longmapsto f(\alpha)$$

BASIS VECTOR FOR 1_I

SO :

$$
\begin{tikzcd}
V^I \times \mathcal{H} \arrow[rr] & & V^I \arrow[dd, "\wr"] \\
\mathrm{Hom}_I(1_I, V) \times \mathrm{Hom}_I(1_I, \underbrace{c\text{-}\mathrm{Ind}_I^G(1_I)}_{\overset{!!}{X}}) \arrow[u, phantom, "\wr"] & \circlearrowright & \\
\mathrm{Hom}_G(X,V) \times \mathrm{Hom}_G(X,X) \arrow[u, phantom, "\wr"] \arrow[rr, "\text{COMP'N}"] & & \mathrm{Hom}_G(X,V)
\end{tikzcd}
$$

FROBENIUS RECIPROCITY

+ SIMILAR ISOMORPHISM-CHASING FOR PRODUCT

ALL THE CONSTRUCTIONS WE'VE DISCUSSED (DESCRIPTION OF $\mathcal{H}$, BASIS, ACTION ON V^I, ...) WORK OVER $\mathbb{Z}$ (REPLACING $\mathbb{C}$) IN PARTICULAR, WE CAN BASE CHANGE AND CONSIDER EVERYTHING W/ $\overline{\mathbb{F}}_p$-COEFFS

COEFFS $= \overline{\mathbb{F}}_p$ FROM NOW ON

IN FACT, IT IS BETTER TO WORK W/ A SLIGHTLY SMALLER COMPACT OPEN SUBGP

$$I \rightsquigarrow I_1 = \text{PRO-}p \text{ SYLOW OF } I$$

"PRO-p-IWAHORI SUBGP" $= \begin{pmatrix} 1+p\mathbb{Z}_p & & \mathbb{Z}_p \\ & \ddots & \\ p\mathbb{Z}_p & & 1+p\mathbb{Z}_p \end{pmatrix}$

// WE LET $\mathcal{H} = \mathcal{H}(G, I_1)_{/\mathbb{F}_p}$

$=$ "PRO-p-IWAHORI-HECKE ALG"

HAS SIMILAR STRUCTURE TO $\mathcal{H}(G,I)_{/\mathbb{C}}$ FROM BEFORE

EG • BASIS $T_w = \mathbb{1}_{I_1 \dot{w} I_1}$ INDEXED BY $w \in \widetilde{W}_1 := N/N \cap I_1$

$\widetilde{W}_1$ IS AN EXT'N OF $\widetilde{W}$ BY $N \cap I / N \cap I_1$

$\cong T \cap I / T \cap I_1$

$\cong \mathbb{T}(\mathbb{F}_p) \leftarrow$ FINITE TORUS

SO HAVE LOTS OF CONTROL OVER STRUCTURE

• THE T_w SATISFY BRAID REL'S AND QUADRATIC RELS

$\hookrightarrow$ FOR $G = SL_2(\mathbb{Q}_p)$

$$T_{s_1}^2 = \underbrace{\left(\boxed{\sum_{x \in \mathbb{F}_p^\times} T_{\left(\begin{smallmatrix} x & \\ & x^{-1}\end{smallmatrix}\right)}}\right)}_{q-1 \text{ TERMS}} T_{s_1} + q \boxed{T_{\left(\begin{smallmatrix} -1 & \\ & -1\end{smallmatrix}\right)}}$$

SPECIALIZE TO T_1 IN $\mathcal{H}(G, I)$

B/C CHAR $= p$

$$= \left(\sum_{x \in \mathbb{F}_p^\times} T_{\left(\begin{smallmatrix} x & \\ & x^{-1}\end{smallmatrix}\right)}\right) T_{s_1}$$

ANALOGOUSLY FOR s_0

• WE UNDERSTAND THE CENTER OF $\mathcal{H}$, BERNSTEIN PRESENTATION, AND <u>ALL</u> SIMPLE MODULES

(VIGNÉRAS, OLLIVIER, ABE)

WHY IS $\mathcal{H} = \mathcal{H}(G, I_1)/_{\overline{\mathbb{F}}_p}$ THE CORRECT OBJECT TO CONSIDER IN CHAR p?

LEMMA SUPPOSE V IS A NONZERO SMOOTH REP OF G (OVER $\overline{\mathbb{F}}_p$). THEN $V^{I_1} \neq 0$

RELIES ON THE FACT THAT I_1 IS PRO-p AND COEFFS $= \overline{\mathbb{F}}_p$

NOT TRUE FOR $\mathbb{C}$-COEFFS! SO EVEN mod p SUPERCUSPIDAL REPS SATISFY $V^{I_1} \neq 0$

UPSHOT V^{I_1} CARRIES MORE INFO IN CHAR p

MOREOVER, V^{I_1} HAS THE STRUCTURE OF AN $\mathcal{H}$-MODULE AND WE HAVE mod p ANALOG OF BOREL-BERNSTEIN THM

THM (OLLIVIER, K.)
SUPPOSE $G = GL_2(\mathbb{Q}_p)$ OR $SL_2(\mathbb{Q}_p)$, AND LET $\mathcal{H} = \mathcal{H}(G, I_1)/_{\overline{\mathbb{F}}_p}$.
THEN

$$\{V \in \mathrm{Rep}_{\overline{\mathbb{F}}_p}(G) : \langle G \cdot V^{I_1} \rangle = V\} = \mathrm{REP}^{I_1}_{\overline{\mathbb{F}}_p}(G) \xrightarrow{\sim} \mathrm{MOD}\text{-}\mathcal{H}$$
$$V \longmapsto V^{I_1}$$

IS AN EQUIV OF CATS, BUT FAILS FOR $GL_2(\mathbb{Q}_{p^f})$ $f \geq 2$.
DIFF THAN $\mathbb{C}$ COEFFS

IN PARTICULAR, SINCE EVERY IRREP IS CONTAINED IN $\mathrm{REP}^{I_1}_{\overline{\mathbb{F}}_p}(G)$, 13

THM GIVES A BIJECTION

$$\left\{ \begin{array}{c} \underline{\text{ALL}} \\ \text{IRREPS } V \\ \text{OF } GL_2(\mathbb{Q}_p) \\ (\text{OR } SL_2(\mathbb{Q}_p)) \end{array} \right\} \Big/ \cong \quad \xleftrightarrow{1:1} \quad \left\{ \begin{array}{c} \text{SIMPLE } \mathcal{H} \\ \text{MODULES} \end{array} \right\} \Big/ \cong$$

AGAIN, NOT TRUE FOR $\mathbb{C}$-COEFFS

(III) LAST TIME: STRUCTURE OF $\mathcal{H}(G,I)_{\mathbb{C}}$ AND $\mathcal{H}(G,I_1)_{\overline{\mathbb{F}}_p}$

TODAY: <u>APPLICATIONS</u>

<u>RECALL</u> $G = GL_n(\mathbb{Q}_p)$ OR $SL_n(\mathbb{Q}_p)$

$$\mathcal{H} = \mathcal{H}(G, I_1)_{/\overline{\mathbb{F}}_p} \quad \longleftarrow \text{NOTATIONAL SWITCH}$$

(A) <u>HECKE MODS</u>

IN THIS GENERALITY, THE CLASSIFICATION OF

$\{$ IRR ADM $\overline{\mathbb{F}}_p$-REPS OF $G\}$ IS STILL UNKNOWN

IF $G \neq GL_2(\mathbb{Q}_p), SL_2(\mathbb{Q}_p)$

<u>BUT</u>

$\{$ SIMPLE MODULES OVER $\mathcal{H}\}$ IS <u>COMPLETELY</u> KNOWN

(ABE, OLLIVIER, VIGNERAS)

<u>EX</u> $G = SL_2(\mathbb{Q}_p)$

$\mathcal{H}$ = GEN'D BY $T_{s_1} = T_{\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}}$, $T_{s_0} = T_{\begin{pmatrix} & -p^{-1} \\ p & \end{pmatrix}}$

AND $T_{\begin{pmatrix} x & \\ & x^{-1} \end{pmatrix}}$ $(x \in \mathbb{F}_p^\times)$

w/

$$\begin{cases} T_w T_{w'} = T_{ww'} & \text{IF } \ell(ww') = \ell(w) + \ell(w') \\ T_{s_i}^2 = \left(\sum_{x \in \mathbb{F}_p^\times} T_{\begin{pmatrix} x & \\ & x^{-1}\end{pmatrix}}\right) T_{s_i} + q\, T_{\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}} & (q := p) \end{cases}$$

THE SIMPLE $\mathcal{H}$-MODULES FALL INTO 4 CATEGORIES

- <u>TRIV CHAR</u>:

  LET $M_{\text{triv}} = \overline{\mathbb{F}}_p v$ BE 1-DIM'L w/ ACTION

  $v \cdot T_{s_1} = 0 \qquad v \cdot T_{s_0} = 0 \qquad v \cdot T_{\begin{pmatrix} x & \\ & x^{-1}\end{pmatrix}} = v$

- SIGN CHAR:

LET $M_{\mathrm{SIGN}} = \overline{\mathbb{F}}_p v$ BE 1-DIM'L W/ ACTION

$$v \cdot T_{s_1} = -v \qquad v \cdot T_{s_0} = -v \qquad v \cdot T_{\left(\begin{smallmatrix} x & \\ & x^{-1}\end{smallmatrix}\right)} = v$$

- "PS" MODULES: FIX $\chi : \mathbb{Q}_p^\times \to \overline{\mathbb{F}}_p^\times$, $\chi \neq 1$, $\delta = \begin{cases} 0 & \chi|_{\mathbb{Z}_p^\times} \neq 1 \\ 1 & \chi|_{\mathbb{Z}_p^\times} = 1 \end{cases}$

LET $M_\chi = \overline{\mathbb{F}}_p v_1 \oplus \overline{\mathbb{F}}_p v_2$ BE 2-DIM'L W/ ACTION

$$v_1 \cdot T_{s_1} = v_2 \qquad v_1 \cdot T_{s_0} = -\delta v_1 \qquad v_1 \cdot T_{\left(\begin{smallmatrix} x & \\ & x^{-1}\end{smallmatrix}\right)} = \chi(x)^{-1} v_1$$

$$v_2 \cdot T_{s_1} = -\delta v_2 \qquad v_2 \cdot T_{s_0} = \chi(p)^{-1} v_1 \qquad v_2 \cdot T_{\left(\begin{smallmatrix} x & \\ & x^{-1}\end{smallmatrix}\right)} = \chi(x) v_2$$

(IF $\chi = 1$, M_χ CONTAINS M_{TRIV}, HAS M_{SIGN} AS QUOT)

- "SUPERSINGULAR" MODULES: FIX $0 \leq r \leq q-1$

LET $M_r = \overline{\mathbb{F}}_p v$ BE 1-DIM'L W/ ACTION

$$r = 0 \qquad v \cdot T_{s_1} = 0 \qquad v \cdot T_{s_0} = -v \qquad v \cdot T_{\left(\begin{smallmatrix} x & \\ & x^{-1}\end{smallmatrix}\right)} = v$$

$$r = q-1 \qquad v \cdot T_{s_1} = -v \qquad v \cdot T_{s_0} = 0 \qquad v \cdot T_{\left(\begin{smallmatrix} x & \\ & x^{-1}\end{smallmatrix}\right)} = v$$

$$0 < r < q-1 \qquad v \cdot T_{s_1} = 0 \qquad v \cdot T_{s_0} = 0 \qquad v \cdot T_{\left(\begin{smallmatrix} x & \\ & x^{-1}\end{smallmatrix}\right)} = x^{-r} v$$

FACT / EXERCISE EVERY SIMPLE $\mathcal{H}$-MODULE IS ISOMORPHIC TO ONE OF M_{TRIV}, M_{SIGN}, M_χ, M_r (AND NO ISOMS AMONG THESE)

RMKS
- THIS DESCRIPTION OF SIMPLE $\mathcal{H}$-MODS HOLDS FOR $SL_2(F)$, EVEN THOUGH WE DON'T KNOW SMOOTH IRREPS

- SIMILAR DESCRIPTION HOLDS FOR $GL_2(F)$ (W/ MORE TWISTS) EXCEPT THAT SUPERSINGULARS ARE 2-DIM'L
- CLASS'N EXTENDS TO GENERAL G

(B) SMOOTH REPS FROM $\mathcal{H}$-MODS

GIVEN A SIMPLE $\mathcal{H}$-MODULE M FOR $GL_2(F)$, A CONSTRUCTION OF PASKUNAS ASSOCIATES TO M A "COEFFICIENT SYSTEM" (OR COSHEAF) ON $\mathcal{T}$, THE BRUHAT-TITS TREE OF G (MORE INFO IN MIHIR'S COURSE)

SO

$$\underset{\text{SIMPLE } \mathcal{H}\text{-MOD}}{M} \rightsquigarrow \underset{\text{COEFF SYSTEM ON } \mathcal{T}}{\mathcal{C}_M} = \{\mathcal{C}_{M,\sigma}\}_{\sigma \text{ EDGE OR VERTEX}}$$

$\longleftrightarrow$ DIAGRAM

AND CAN LOOK AT

$$H_0(\mathcal{T}, \mathcal{C}_M) = \operatorname{coker}\Big(\underset{= I \rtimes \omega^{\mathbb{Z}}}{\text{c-Ind}_{N(I)}^{G}} (\mathcal{C}_{M,c} \otimes \mu_{-1}) \xrightarrow{\partial} \text{c-Ind}_{KZ}^{G} (\mathcal{C}_{M,x_0}) \Big) \quad \text{SMOOTH } G \text{ REP}$$

EDGE (c), VERTEX (x_0)

$$\partial([\![g, v]\!]) = [g, r_{x_0}^{c}(v)] - [g\omega, r_{x_0}^{c}(\omega^{-1} v)]$$

WON'T GO INTO CONSTRUCTION OF $\mathcal{C}_M$, CONSULT PASKUNAS' PAPER

THM (PASKUNAS) IF M IS A SUPERSINGULAR $\mathcal{H}$-MOD, THEN $H_0(\Upsilon, \mathcal{C}_M)$ ADMITS AN IRR ADM QUOTIENT V. MOREOVER, THE $\mathcal{H}$-MOD V^{I_1} CONTAINS M, AND THUS V IS SUPERSINGULAR

RMK BY USING GLOBAL CONSIDERATIONS, THESE REPS V WILL **NOT** CONTRIBUTE TO A LOCAL LANGLANDS CORR (WRONG SERRE WEIGHTS)

- DIFF'T STRATEGY THAN MILITR'S COURSE: THIS PROVES SUPERSINGULARITY BY SHOWING $M \subset V^{I_1}$, NOT BY COUNTING $\dim(V^{I_1})$

// WHEN $G = GL_2(\mathbb{Q}_p)$, WE CAN GET EVEN MORE INFO:

THM (OLLIVIER) LET $G = GL_2(\mathbb{Q}_p)$, AND LET V BE A G-REP w/ $\langle G \cdot V^{I_1} \rangle = V$. LET $\mathcal{C}_V$ DENOTE THE COEFF SYSTEM ASS'D TO THE DIAGRAM

$$\left(V^{I_1} \hookrightarrow \langle K \cdot V^{I_1} \rangle \right)$$

THEN $H_0(\Upsilon, \mathcal{C}_V) \cong V$, IE V ADMITS A G-EQUIV RESOLUTION

$$0 \to \text{c-Ind}_{\omega^{\mathbb{Z}} \times I}^{G} \left(V^{I_1} \otimes \mu_{-1} \right) \to \text{c-Ind}_{KZ}^{G} \left(\langle K \cdot V^{I_1} \rangle \right) \longrightarrow V \longrightarrow 0$$

RMKS
- PF USES EQUIV OF CATS
- RESOLUTION ABOVE IS A "STANDARD PRESENTATION" OF V IN THE SENSE OF COLMEZ AND WAS USED IN THE CONSTRUCTION OF THE p-ADIC LLC

(c) (φ, Γ)-MODULES FROM $\mathcal{H}$-MODS

SUPPOSE $G = GL_n(\mathbb{Q}_p)$, AND LET M DENOTE AN $\mathcal{H}$-MODULE. AGAIN USING COEFF SYSTEMS, GROSSE-KLONNE ASSOCIATES TO M A COEFF SYSTEM ON Υ^+ (HALF TREE)

$$\underset{\mathcal{H}\text{-MOD}}{M} \rightsquigarrow \underset{\text{COEFF SYSTEM ON } \Upsilon^+}{D_M}$$

AND CAN LOOK AT $H_0(\Upsilon^+, D_M)$. THIS IS NO LONGER A G-REP, BUT IT CAN BE ENDOWED W/ STRUCTURE OF A $(\varphi, \underset{\mathbb{Z}_p^\times}{\underset{\shortparallel}{\Gamma}})$-MOD

THM (GROSSE-KLONNE) LET $G = GL_n(\mathbb{Q}_p)$. THE MAP

$$M \longmapsto D(H_0(\Upsilon^+, D_M))$$

FONTAINE'S EQUIV OF CATS

INDUCES A BIJECTION

$$\left\{ \begin{array}{l} M \text{ SSING} \\ \mathcal{H}\text{-MOD OF DIM } n \end{array} \right\} \Big/ \cong \quad \xleftrightarrow{1:1} \quad \left\{ \begin{array}{l} n\text{-DIM'L IRREPS} \\ \text{OF } \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \end{array} \right\} \Big/ \cong$$

THIS GIVES A FUNCTORIAL LLC FOR HECKE MODS

NOW SPECIALIZE TO $n = 2$

THM (GROSS-KLÖNNE) LET $G = GL_2(\mathbb{Q}_p)$. THEN THE COMPOSITE

$$\begin{array}{ccccc} V & \longmapsto & V^{I_1} & \longmapsto & D(H_0(\mathcal{T}^+, \mathcal{D}_{V^{I_1}})) \\ \text{IRREP OF } G & & \text{SIMPLE } \mathcal{H}\text{-MOD} & & \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)\text{-REP} \end{array}$$

AGREES w/ mod p LLC OF COLMEZ

(D) EXTENSIONS OF IRREPS OF $GL_2(\mathbb{Q}_p)$ (PREVIEW OF DERIVED THINGS)

NOW TAKE $G = GL_2(\mathbb{Q}_p)$ OR $SL_2(\mathbb{Q}_p)$, SO THAT WE HAVE AN EQUIVALENCE

$$\mathrm{REP}^{I_1}_{\overline{\mathbb{F}}_p}(G) \xrightarrow{\sim} \mathrm{MOD}-\mathcal{H}$$

SUPPOSE V, V' ARE TWO REPS OF G, AND ASSUME $\langle G \cdot V^{I_1} \rangle = V$.

LET $V_1' = \langle G \cdot (V')^{I_1} \rangle \in \mathrm{REP}^{I_1}_{\overline{\mathbb{F}}_p}(G)$. THEN

$$\begin{aligned} \mathrm{HOM}_G(V, V') &\cong \mathrm{HOM}_G(V, V_1') \overset{\text{EQUIV}}{\cong} \mathrm{HOM}_{\mathcal{H}}(V^{I_1}, (V_1')^{I_1}) \\ &\cong \mathrm{HOM}_{\mathcal{H}}(V^{I_1}, (V')^{I_1}) \end{aligned}$$

$$\begin{tikzcd} \mathrm{REP}_{\overline{\mathbb{F}}_p}(G) \arrow[r, "V' \mapsto (V')^{I_1}"] \arrow[rr, bend right, "\mathrm{HOM}_G(V, -)"'] & \mathrm{MOD}-\mathcal{H} \arrow[r, "\mathrm{HOM}_{\mathcal{H}}(V^{I_1}, -)"] & \overline{\mathbb{F}}_p\text{-VEC} \end{tikzcd}$$

IF $J \in \mathrm{REP}_{\overline{\mathbb{F}}_p}(G)$ IS INJECTIVE, THEN $J_1 \in \mathrm{REP}^{I_1}_{\overline{\mathbb{F}}_p}(G)$ IS ALSO INJECTIVE $\Longrightarrow$ CAN CONSTRUCT A GROTHENDIECK SPECTRAL SEQUENCE

$$\mathrm{Ext}^i_{\mathcal{H}}\left(V^{I_1}, H^j(I_1, V')\right) \Longrightarrow \mathrm{Ext}^{i+j}_G(V, V')$$

ESSENTIALLY A NON-NORMAL HOCHSCHILD-SERRE SS

(IF V' IS A SMOOTH G-REP, $(V')^{I_1} = H^0(I_1, V')$ IS AN $\mathcal{H}$-MOD, AND SO IS $H^j(I_1, V')$ FOR $j > 0$)

IMPOSSIBLE TO WRITE DOWN INJ RES

SO, TO COMPUTE $\mathrm{Ext}^n_G(V, V')$ (HARD), WE JUST NEED TO COMPUTE $H^j(I_1, V')$ FOR ALL j AND $\mathrm{Ext}^i_{\mathcal{H}}(V, -)$ (LESS HARD) (USES COMB'L STRUCTURE OF $\mathcal{H}$, CAN COMPUTE INJ RES'S)

PROP (PASKUNAS, K.) IF V' IS AN IRREP OF $G = GL_2(\mathbb{Q}_p)$ OR $SL_2(\mathbb{Q}_p)$, THEN WE CAN CALCULATE $H^i(I_1, V')$ AS AN $\mathcal{H}$-MODULE FOR ALL i.

<u>EX</u> SUPPOSE $G = SL_2(\mathbb{Q}_p)$ $(p > 3)$

- IF $V = \overline{\mathbb{F}}_p$ DENOTES THE TRIVIAL G-REP,

DIMS		
1	$H^0(I_1, \overline{\mathbb{F}}_p) \cong M_{\mathrm{triv}}$	
2	$H^1(I_1, \overline{\mathbb{F}}_p) \cong M_{\alpha}$	PS
2	$H^2(I_1, \overline{\mathbb{F}}_p) \cong M_{\bar{\alpha}}$	PS
1	$H^3(I_1, \overline{\mathbb{F}}_p) \cong M_{\mathrm{triv}}$	

$$\alpha : \mathbb{Q}_p^\times \longrightarrow \overline{\mathbb{F}}_p^\times$$

$$\begin{pmatrix} x & \\ & x^{-1} \end{pmatrix} \longmapsto x^2 |x^2|_p \mod p$$

OTHER COH IS 0

- IF $V = \mathrm{IND}_B^G(\chi)$ IS A PS REP OF G,

DIMS		
2	$H^0(I_1, V) = M_\chi$	
4	$H^1(I_1, V) = M_\chi \text{ — } M^\vee_{\chi^{-1}\bar{\alpha}}$	POTENTIALLY NONSPLIT EXT'N
2	$H^2(I_1, V) = M^\vee_{\chi^{-1}\bar{\alpha}}$	ETC.

THM (PASKUNAS, NADIMPALLI, K.)

USING THESE CALCULATIONS, IT IS POSSIBLE TO CALCULATE

$$\mathrm{EXT}^n_G(V, V')$$

FOR $G = GL_2(\mathbb{Q}_p)$, ALL n AND ALL IRREPS V, V'

EX FOR $G = SL_2(\mathbb{Q}_p)$

		DIMS ($n = 0, 1, 2, \ldots$)
$\chi^2 \neq \bar{\alpha}$	$\mathrm{EXT}^n_G(\mathrm{IND}_B^G(\chi), \mathrm{IND}_B^G(\chi))$	$1, 2, 1, 0, 0, 0 \ldots$
	$\mathrm{EXT}^n_G(\mathrm{IND}_B^G(\chi), \mathrm{IND}_B^G(\chi^{-1}\bar{\alpha}))$	$0, 1, 2, 1, 0, 0, \ldots$
	OTHER $\mathrm{IND}_B^G(\psi)$ GIVE 0	
$\chi^2 = \bar{\alpha}$	$\mathrm{EXT}^n_G(\mathrm{IND}_B^G(\chi), \mathrm{IND}_B^G(\chi))$	$1, 3, 3, 1, 0, 0, \ldots$

(EXCEPT IF $\chi(x) = x|x|_p \bmod p$)

IV LAST TIME: APPLICATIONS

TODAY: DERIVED STRUCTURE IN CHAR p

NOTATION $G = GL_n(\mathbb{Q}_p)$ BUT CAN TAKE ANY CONNECTED REDUCTIVE GP$/\mathbb{Q}_p$

$$I_1 = \begin{pmatrix} 1+p\mathbb{Z}_p & & \mathbb{Z}_p \\ & \ddots & \\ p\mathbb{Z}_p & & 1+p\mathbb{Z}_p \end{pmatrix}$$

PRO-p-IWAHORI SUBGP

$\mathcal{H} = \mathcal{H}(G, I_1)_{/\overline{\mathbb{F}}_p}$ = PRO-p-IWAHORI HECKE ALG

$$= \left\{ f : G \to \overline{\mathbb{F}}_p : \begin{array}{l} \cdot\ f \text{ CPCT SUPP} \\ \cdot\ f(igi') = f(g) \\ \quad \forall i, i' \in I_1,\ g \in G \end{array} \right\}$$

w/ CONVOLUTION PRODUCT

RECALL: THE FUNCTOR

$$\{V \in \mathrm{Rep}_{\overline{\mathbb{F}}_p}(G) : \langle G \cdot V^{I_1} \rangle = V\} = \mathrm{Rep}^{I_1}_{\overline{\mathbb{F}}_p}(G) \longrightarrow \mathrm{Mod}\text{-}\mathcal{H}$$

$$V \longmapsto V^{I_1}$$

GIVES AN EQUIV OF AB CATS FOR $G = GL_2(\mathbb{Q}_p)$ OR $G = SL_2(\mathbb{Q}_p)$, BUT FAILS IN GENERAL

REASON: • IN CALC'N OF $(M \otimes_{\mathcal{H}} X)^{I_1}$, CERTAIN CHARACTER SUMS w/ WITT VECTORS ARE NONZERO $\Leftrightarrow F = \mathbb{Q}_p$ (BOTH OLLIVIER + BREUIL)

• HAVE CLASS'N RESULT FOR MODULES OVER $\overline{\mathbb{F}}_p[\![\mathcal{O}_F]\!]$ $\Leftrightarrow F = \mathbb{Q}_p$ (PASKUNAS)

PROBLEM: SINCE I_1 IS PRO-p AND $\mathrm{char}(\overline{\mathbb{F}}_p) = p$, THE FUNCTOR

$$V \longmapsto V^{I_1}$$ IS NOT EXACT

SO IT MAKES SENSE TO DERIVE THIS FUNCTOR AND USE DERIVED CATEGORIES

1) REP THEORY SIDE: $\mathrm{Rep}_{\overline{\mathbb{F}}_p}(G)$ IS A GROTHENDIECK ABELIAN CATEGORY W/ ENOUGH INJECTIVES AND WHICH ADMITS ALL RIGHT DERIVED FUNCTORS

NO I_1

$\rightsquigarrow$ WE CAN CONSTRUCT $D(G) := D(\mathrm{Rep}_{\overline{\mathbb{F}}_p}(G))$

UNBOUNDED

THE CATEGORY $D(G)$ REMEDIES (SOME OF) THE PROBLEMS OF $\mathrm{Rep}_{\overline{\mathbb{F}}_p}(G)$:

EX SUPPOSE V IS A REP OF G. THEN DEFINE

$$V^\vee := \text{SMOOTH VECTORS IN } V^* = \mathrm{Hom}_{\overline{\mathbb{F}}_p}(G, \overline{\mathbb{F}}_p),$$

$$= \varinjlim_{\substack{J \leq G \\ \text{compact open}}} \mathrm{Hom}_J(G, \overline{\mathbb{F}}_p)$$ "CONTRAGREDIENT"

- OVER $\mathbb{C}$, $V \longmapsto V^\vee$ IS EXACT AND SATISFIES $V^{\vee\vee} = V$ FOR ADMISSIBLE V
- OVER $\overline{\mathbb{F}}_p$, IF V IS IRRED, ADM, AND $\dim(V) = \infty$, THEN $V^\vee = 0$!

THIS CAN BE FIXED IN $D(G)$

THM (KOHLHAASE, SCHNEIDER–SORENSEN)

$\exists$ A CONTRAVARIANT DUALITY FUNCTOR

$$S : D(G) \longrightarrow D(G)$$
$$V^\bullet \longmapsto \varinjlim_J \mathrm{RHom}_J(V^\bullet, \overline{\mathbb{F}}_p[0])$$

WHICH SATISFIES

- $S(V) = V^\vee$ IF V IS FINITE DIM'L
- $S^2 = \mathrm{id}$ ON $D_{\mathrm{adm}}(G)$

COMPLEXES W/ ADM COH OBJECTS

THE FUNCTOR S HAS BEEN CALCULATED IN MANY EXAMPLES

EX $\quad G = SL_2(\mathbb{Q}_p)$

- IF $V = St$, $\quad S(St) = \mathcal{E}[-1]$, WHERE
$$0 \to 1_G \to \mathcal{E} \to \mathrm{Ind}_B^G(\alpha) \to 0 \quad \text{NONSPLIT}$$
- IF $V = \mathrm{Ind}_B^G(\chi)$, $\quad S(\mathrm{Ind}_B^G(\chi)) = \mathrm{Ind}_B^G(\chi^{-1}\alpha)[-1]$

② HECKE ALG SIDE: NEED TO ENHANCE MOD-$\mathcal{H}$

CHOOSE AN INJECTIVE RES'N

$$0 \to X \to I^0 \to I^1 \to \cdots$$

AND DEFINE

$$\mathcal{H}^\bullet = \mathrm{Hom}_G^\bullet(I^\bullet, I^\bullet) \cong \mathrm{RHom}_G(X, X) \quad \text{IN } D(\mathbb{F}_p)$$

"HECKE DIFFERENTIAL GRADED ALGEBRA (DGA)" INVESTIGATED BY SCHNEIDER

WE CAN THEN DEFINE THE CATEGORY

$$D(\mathrm{DGMOD}\text{-}\mathcal{H}^\bullet)$$

N.B. $\mathcal{H}^\bullet$ DEPENDS ON CHOICE OF $\mathcal{I}^\bullet$, BUT IT IS WELL-DEFINED UP TO QUASI-ISOM, AND $D(\text{DGMOD-}\mathcal{H}^\bullet)$ IS WELL-DEFINED UP TO NAT. EQUIV.

THM (SCHNEIDER) SUPPOSE I_1 IS TORSION-FREE. THEN

$$D(G) \xrightarrow{\sim} D(\text{DGMOD-}\mathcal{H}^\bullet)$$

$$V^\bullet \longmapsto \text{HOM}_G^\bullet(\mathcal{I}^\bullet, i(V^\bullet))$$

(K-)INJ RES OF $V^\bullet$

$$\cong R\text{HOM}_G(X, V^\bullet)$$
$$\cong RH^0(I_1, V^\bullet)$$

IS AN EQUIV OF TRIANGULATED CATS

ISSUE: IT IS HARD (IMPOSSIBLE?) TO WRITE DOWN EXPLICIT INJ RES $X \longrightarrow \mathcal{I}^\bullet \Rightarrow$ HARD TO DESCRIBE STRUCTURE OF $\mathcal{H}^\bullet$ SO HARD TO CALCULATE W/ $\mathcal{H}^\bullet$ DIRECTLY

INSTEAD: LOOK AT COHOMOLOGY OF $\mathcal{H}^\bullet$

$$h^i(\mathcal{H}^\bullet) = h^i(R\text{HOM}_G(X,X)) \cong \text{EXT}_G^i(X,X)$$

DEF THE PRO-p - IWAHORI -EXT ALGEBRA IS

$$E := \bigoplus_{i \in \mathbb{Z}} E^i := \bigoplus_{i\in\mathbb{Z}} \text{EXT}_G^i(X,X)$$

PROPERTIES

- E IS A GRADED ALGEBRA W/ YONEDA PRODUCT
- $E^0 = \mathcal{H}$, AND EACH E^i IS AN $\mathcal{H}$-$\mathcal{H}$ BIMODULE

- WE HAVE

$$E^i = \mathrm{EXT}^i_G(X, X)$$

FROB RECIP

$$\cong \mathrm{EXT}^i_{I_1}(1_{I_1}, X|_{I_1})$$

$$\cong H^i(I_1, X)$$

MACKEY

$$\cong \bigoplus_{w \in \widetilde{W}_1} H^i(I_1, \mathrm{IND}_{I_w}^{I_1}(1))$$

$I_w = I_1 \cap w^{-1} I_1 w$

SHAPIRO

$$\cong \bigoplus_{w \in \widetilde{W}_1} H^i(I_w, \mathbb{F}_p)$$

$$\text{c-IND}_{I_1}^G(1)\big|_{I_1}$$

SUPP $\subset I_1 w I_1$

$$\cong \bigoplus_{w \in \widetilde{W}_1} \text{c-IND}_{I_1}^{I_1 w I_1}(1)$$

$$\cong \bigoplus_{w \in \widetilde{W}_1} \mathrm{IND}_{I_1 \cap w^{-1} I_1 w}^{I_1}(1)$$

- IF I_1 IS TORSION-FREE, THEN IT IS A POINCARÉ GP OF DIM $d := \dim_{\mathbb{Q}_p}(G)$.

$$\Rightarrow \quad H^i(I_w, X) = 0 \quad \text{IF} \quad i > d$$

$$\Rightarrow \quad E^i \text{ CONCENTRATED IN DEGS } 0, 1, \ldots, d.$$

WORKS FOR ANY SEMISIMPLE SIMPLY CONN GP

- IF $G = SL_n(\mathbb{Q}_p)$ AND I_1 IS TORS-FREE,

THEN $d = n^2 - 1$, AND

TOP DEGREE

$$E^{n^2-1} \cong M_{\mathrm{TRIV}} \oplus (\text{BIG SUPERSINGULAR MODULE})$$

(AS EITHER LEFT OR RIGHT $\mathcal{H}$-MODS)

// PRODUCT STRUCTURE ON E CAN BE DESCRIBED EXPLICITLY. FOR EX:

PROP (OLLIVIER-SCHNEIDER) SUPPOSE $v, w \in \widetilde{W}_1$ SATISFY

$$\ell(vw) = \ell(v) + \ell(w)$$

LET $\alpha \in H^i\left(I_1, \operatorname{IND}_{I_v}^{I_1}(1)\right)$, $\beta \in H^j\left(I_1, \operatorname{IND}_{I_w}^{I_1}(1)\right)$. THEN

$$\alpha \circledast \beta = \underbrace{\alpha \cdot T_w}_{\in H^i\left(I_1, \operatorname{IND}_{I_{vw}}^{I_1}(1)\right)} \smile \underbrace{T_v \cdot \beta}_{\in H^j\left(I_1, \operatorname{IND}_{I_{vw}}^{I_1}(1)\right)}$$

PRODUCT IN E

$$\in H^{i+j}\left(I_1, \operatorname{IND}_{I_{vw}}^{I_1}(1)\right)$$

FOLLOWS FROM EXPLICIT TECHNICAL CALCULATION W/ GROUP COHOMOLOGY

O-S ALSO DESCRIBE A CERTAIN DUALITY OPERATION RELATING E^i TO $E^{d-i,\vee}$

// **Q** HOW DO WE UNDERSTAND SCHNEIDER'S EQUIVALENCE?

CAN PASS TO COHOMOLOGY:

$$\begin{array}{ccccc} \mathrm{RHom}_G(X, V^\bullet) & \overset{\oplus h^i(-)}{\rightsquigarrow} & \bigoplus_{i\in\mathbb{Z}} \mathrm{Ext}^i_G(X, V^\bullet) & = & \bigoplus_{i\in\mathbb{Z}} \mathbb{H}^i(I_1, V^\bullet) \\ \circlearrowleft & & \circlearrowleft & & \circlearrowleft \\ \mathcal{H}^\bullet & & \bigoplus_{i\in\mathbb{Z}} h^i(\mathcal{H}^\bullet) & & \bigoplus_{i\in\mathbb{Z}} E^i =: E \end{array}$$

IE, WE CAN CONSIDER THE GRADED ACTION OF E ON THE HYPERCOHOMOLOGY $\bigoplus_{i\in\mathbb{Z}} \mathbb{H}^i(I_1, V^\bullet)$

<u>SPECIAL CASE</u> TAKE $V^\bullet = V[0]$ FOR A SMOOTH REP V

THEN THE GRP COH

$\dim_{\mathbb{Q}_p}(G)$

FINITE COH DIM

$$\bigoplus_{i=0}^{d} H^i(I_1, V)$$

IS A GRADED RIGHT E-MODULE. BY EXAMINING THE STRUCTURE OF E^i, O-S CAN DEDUCE PROPERTIES OF $H^i(I_1, V)$:

<u>PROP</u> (OLLIVIER-SCHNEIDER)

- SUPPOSE $G = SL_2(F)$ AND I_1 IS TORS FREE.

LET V BE A SMOOTH IRREP OF G. THEN

TOP DEGREE

$$H^{3[F:\mathbb{Q}_p]}(I_1, V) = \begin{cases} \overline{\mathbb{F}}_p & V = 1_G \\ 0 & V \neq 1_G \end{cases}$$

- IF $G = SL_2(\mathbb{Q}_p)$ $(p > 3)$ AND $i \in [0,3]$ IS FIXED, THEN

V IS A SUPERSINGULAR G-REP $\iff$ $H^i(I_1, V)$ IS A SUPERSINGULAR $\mathcal{H}$-MOD

EARLIER WORK (K.) PROVED THIS USING OTHER METHODS;
O-S USE STRUCTURE OF E, AND DO NOT REQUIRE CLASS'N OF IRREPS BUT DO REQUIRE EQUIV OF CATS FOR SECOND PT

THE STORY CONTINUES: THERE ARE MANY ASPECTS OF THE ALGEBRA E AND THE ACTION $E \circlearrowright \bigoplus H^i(I_1, V)$ WHICH ARE UNEXPLORED

ONGOING: COMBINE THIS W/ DESCRIPTION OF $E = \bigoplus_i E^i$ FOR $SL_2(\mathbb{Q}_p)$ (OLLIVIER-SCHNEIDER) TO CALCULATE GRADED ACTION OF E ON $\bigoplus_{i=0}^{3} H^i(I_1, V)$ $\forall$ IRRED V (K.)

OPEN PROBLEM (HARD!) CALCULATE THE STRUCTURE OF $E = \bigoplus_i E^i$ FOR $G = D^\times$, $D/\mathbb{Q}_p$ CENTRAL DIV. ALG OF DIM d^2.
USE THIS TO DESCRIBE ACTION ON $\bigoplus_i H^i(1+\mathfrak{m}_D, V)$ I_1
FOR ANY IRREP V OF $D^\times$ (START W/ $V = \overline{\mathbb{F}}_p$)

EVEN UNDERSTANDING $i=2$ CASE IS INTERESTING!