§1.1

Let F be a number field, i.e., F is a finite extension of $\mathbb{Q}$.

Let v denote an absolute value of F (finite or infinite). We let $\mathcal{O}_v$ be the ring of integers of F_v and $\mathcal{U}_v$ its group of units.

We define the adèle and idèle groups of F as follows. As sets they are:

$$\mathbb{A}_F = \{ x \in \prod_v F_v \;/\; x_v \in \mathcal{O}_v \quad \forall' v \}$$

$$\mathbb{I}_F = \{ x \in \prod_v F_v^* \;/\; x_v \in \mathcal{U}_v \;\forall' v \}.$$

There is a natural ring structure on $\mathbb{A}_F$ that arises from the ring (field) structure on each factor. The group of multiplicative units in $\mathbb{A}_F$ is $\mathbb{I}_F$.

The points is that $\mathbb{A}_F$ and $\mathbb{I}_F$ can be equipped with locally compact topologies making them locally compact groups (the adeles will become a locally compact ring). They are both specific cases of the restricted product topology.

RESTRICTED PRODUCTS:

Let $\{G_i\}$ $i \in I$ be a family of locally compact (topological) groups, and $\{H_i \leq G_i\}_{i \in I}$ a family of compact subgroups. Define

$$G = \prod_{i \in I}{}' G_i = \{ x = (x_i)_{i \in I} \mid x_i \in H_i \ \forall' i \}.$$

The open subsets of G are those of the form

$$V = \prod_{i \in S} V_i \times \prod_{i \notin S} H_i \qquad (V_i \leq G_i \text{ open for } i \in S)$$

Remarks 1. G is a locally compact topological group. The projections $\pi_i : G \longrightarrow G_i$ are continuous.

2. Replacing finitely many subgroups H_i by subgroups H_i' does not change either the underlying set G or the underlying topology.

3. When $I = M_F$ the set of places of a number field F, $G_v = F_v$ and $H_v = \mathcal{O}_v$ we obtain $\mathbb{A}_F$. When $G_v = F_v^\times$ and $H_v = \mathcal{U}_v$ we obtain $\mathbb{I}_F$.

Note $F_v^\times = GL_1(F_v)$, $\mathcal{U}_v = GL_1(\mathcal{O}_v)$. so

$$\mathbb{I}_F = GL_1(\mathbb{A}_F).$$

<u>WARNING</u>: The topology on $\mathbb{I}_F$ is <u>not</u> the topology induced from the inclusion into $\mathbb{A}_F$

$\mathbb{I}_F = GL_1(\mathbb{A}_F)$

WARNING: The topology on $\mathbb{I}_F$ is not the topology induced from the inclusion into $\mathbb{A}_F$.

4. When $G_v = GL_n(F_v)$ / $SL_n(F_v)$ and $H_v = GL_n(\mathcal{O}_v)$ / $SL_n(\mathcal{O}_v)$ we get the locally compact groups $GL_n(\mathbb{A}_F)$ and $SL_n(\mathbb{A}_F)$.

5. We can also take $F = \mathbb{F}_q(t)$ and M_F to be the places of F.

6. The restricted product has the following alternate description:

Let $S \subseteq I$ be a finite set. Define

$$G_S = \prod_{i \in S} G_i \times \prod_{i \notin S} H_i \; ; \; G^S = \{x \in G \mid x_v = 1 \ \forall\, v \in S\}$$

Then G_S is a locally compact topological group and we can check that the direct limit $\varinjlim_S G_S$ exists in the category of locally compact groups and is isomorphic (algebraically and topologically) to the restricted product G defined above.

iPad 10:59 AM 58%
Maassforms

Exercise 1.1.1: Construct the direct limit in the category of locally compact groups and prove the statement above.

Exercise 1.1.2: The topology on $\mathbb{I}_F$ is the topology on $GL_1(\mathbb{A}_F)$ viewed as an algebraic group: Recall that we view $GL_1(\mathbb{A}_F)$ as a closed subgroup of $GL_2(\mathbb{A}_F) \hookrightarrow M_2(\mathbb{A}_F)$ via the map
$$g \longrightarrow \begin{pmatrix} g & \\ & g^{-1} \end{pmatrix}.$$

§1.2. The Adèles

The field F embeds diagonally into $\mathbb{A}_F$:
$$x \xrightarrow{i} (x, x, \ldots, x, \ldots)$$

For $x \in F$ we have the product formula:
$$\|x\| := \prod_{v \in M_F} \|x\|_v = 1$$

This shows that $i(F)$ is discrete in $\mathbb{A}_F$. Indeed, let $V = \prod_{v \mid \infty} (-1, 1) \times \prod_{v \nmid \infty} \mathcal{O}_v$. Then $F \cap V = \{0\}$.

If $\alpha \in F$ then $(\alpha + V) \cap F = \{\alpha\}$.

iPad 4:45 PM 27%

If $\alpha \in F$, then $(\alpha + V) \cap F = \{\alpha\}$.

From now on we write $\mathbb{A}$ for $\mathbb{A}_F$ to make the notation less cumbersome.

<u>Theorem</u> 1.2.1 $\mathbb{A} = F + \mathbb{A}_{S_\infty}$

(Recall that $\mathbb{A}_{S_\infty} = \{ x \in \mathbb{A} \mid \|x_v\|_v \leq 1 \ \forall v \notin S_\infty \}$.

Here $S_\infty = \{ v \in M_F \mid v | \infty \}$.

(ii) $\mathbb{A}/F$ is compact.

Proof: Let $x \in \mathbb{A}$. Choose $m \in \mathbb{Z}$ such that $(mx)_v \in \mathcal{O}_v \ \forall v \in M_F$. Let $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ be the prime factorisation of m. By the Chinese Remainder Theorem we can find $a \in F$ such that $mx - a \equiv 0 \pmod{p_i^{\alpha_i} \mathcal{O}_v} \ \forall 1 \leq i \leq r$.

Hence, $x - a/m \in \mathcal{O}_v \ \forall v.$, i.e. $x - a/m \in \mathbb{A}_{S_\infty}$

(ii) Choose a $\mathbb{Z}$-basis $\{v_1, \ldots, v_n\}$ of $\mathcal{O}_F$.

Let $B = \{ x \in \prod_{v|\infty} F_v \mid x = \sum_{i=1}^{n} a_i v_i, \ 0 \leq a_i < 1 \}$

Given $x - a/m$ as above we can find $b \in \mathcal{O}_F$ such that $x - a/m - b \in B \times \prod_{v \nmid \infty} \mathcal{O}_v = D$.

Find $\alpha \in F$ such $x - \alpha \in D$. Since D is

If $\alpha \in F$, then $(\alpha + V) \cap F = \{\alpha\}$.

From now on we write $\mathbb{A}$ for $\mathbb{A}_F$ to make the notation less cumbersome.

<u>Theorem</u> 1.2.1 $\mathbb{A} = F + \mathbb{A}_{S_\infty}$

(Recall that $\mathbb{A}_{S_\infty} = \{ x \in \mathbb{A} \mid \|x_v\|_v \leq 1 \ \forall v \notin S_\infty \}$.

Here $S_\infty = \{ v \in M_F \mid v | \infty \}$.

(ii) $\mathbb{A}/F$ is compact.

Proof: Let $x \in \mathbb{A}$. Choose $m \in \mathbb{Z}$ such that $(mx)_v \in \mathcal{O}_v \ \forall v \in M_F$. Let $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ be the prime factorisation of m. By the Chinese Remainder Theorem we can find $a \in F$ such that $mx - a \equiv 0 \pmod{p_i^{\alpha_i} \mathcal{O}_v} \ \forall 1 \leq i \leq r$.

Hence, $x - a/m \in \mathcal{O}_v \ \forall v.$, i.e. $x - a/m \in \mathbb{A}_{S_\infty}$

(ii) Choose a $\mathbb{Z}$-basis $\{v_1, \ldots, v_n\}$ of $\mathcal{O}_F$.

Let $B = \{ x \in \prod_{v|\infty} F_v \mid x = \sum_{i=1}^{n} a_i v_i, \ 0 \leq a_i < 1 \}$

Given $x - a/m$ as above we can find $b \in \mathcal{O}_F$ such that $x - a/m - b \in B \times \prod_{v \nmid \infty} \mathcal{O}_v = D$.

iPad 4:48 PM 29%
Maassforms

Given $x - a/m$ as above we can find $b \in \mathcal{O}_F$ such that $x - a/m - b \in B \times \prod_{v \nmid \infty} \mathcal{O}_v = D$.

find $\alpha \in F$ such $x - \alpha \in D$. Since D is contained in a compact set, we see that $\mathbb{A}/F$ is compact.

Remark: The set D is a fundamental domain for the action of F on $\mathbb{A}$.

Let $\mathbb{I}^1 = \{ x \in \mathbb{I} \mid \|x\| = 1 \}$. Because of the product formula we know that the diagonal inclusion $i(F^\times)$ in $\mathbb{I}$ is discrete. One also checks that $i(F^\times)$ is discrete in $\mathbb{I}^1$. Note that $\mathbb{I} \simeq \mathbb{R}^+ \mathbb{I}^1$. $\left(x \longrightarrow \|x\| \cdot \frac{x}{\|x\|} \right)$.

Theorem 1.2.2: $\mathbb{I}^1/F^*$ is compact.

We defer the proof the theorem.

Alternate notation for the restricted product. Instead of writing $G = \prod_v' G_v$, we also write $\prod_{i \in I} (G_i, H_i)$ sometimes.

§ 1.3 Haar Measures.

iPad 4:48 PM 29%

Alternate notation for the restricted product

Instead of writing $G = \prod_v' G_v$, we also write $\prod_{i \in I} (G_i, H_i)$ sometimes.

§ 1.3 Haar Measures.

If $G = \prod (G_i, H_i)$, there is a natural choice of Haar Measure on G. If $V = \prod_{i \in S} V_i \times \prod_{i \notin S} H_i$ is an open set, we define a measure $\mu(V)$ as follows:

(1) $\mu(H_i) = 1 \quad \forall i \in I$

(2) $\mu(V) = \prod_{i \in S} \mu(V_i)$.

Examples: (1) $G = \mathbb{A}$: Recall that each local field F_v carries a measure μ_v defined by $\mu(\mathcal{O}_v) = 1$. If ϖ_v is a uniformising element for F_v, $\mu(\varpi_v^n \mathcal{O}_v) = \|\varpi_v\|_v^n$. When v / ∞ we make the following choices:

(i) If $F_v \simeq \mathbb{R}$, define $\mu_v = \mu_{\mathbb{R}}$, the standard Euclidean (or Lebesgue) measure on F_v.

(ii) If $F_v \simeq \mathbb{C}$, define $\mu_v = 2\mu_{\mathbb{C}}$, where $\mu_{\mathbb{C}}$ is the usual Lebesgue measure on $\mathbb{C} \simeq \mathbb{R} \times \mathbb{R}$.

(2) A Haar measure on $\mathbb{I}$ can be defined by

$$d^\times\mu_v = m_v \frac{d\mu_v}{\|x_v\|_v}, \quad \text{where } m_v = \begin{cases} 1 & \text{if } v \mid \infty \\ \left(1-\frac{1}{q_v}\right)^{-1} & \text{if } v \nmid \infty. \end{cases}$$

Here q_v is the cardinality of the residue field. It is easy to see that $d^\times\mu_v(\mathcal{U}_v) = 1$

(3) For $G = GL_2(\mathbb{A}_\mathbb{Q})$ we specify the Haar measure as follows: When $v \nmid \infty$ we set

$\mu(K_v) = 1$ where $K_v = GL_2(\mathbb{Z}_p)$, p being the prime corresponding to v.

When $v \mid \infty$, we use the Iwasawa decomposition. Every $g \in GL_2(\mathbb{R}^+)$ can be written as

$$g = \begin{pmatrix} u & \\ & u \end{pmatrix} \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix} k_\theta, \quad k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

[also called that NAK or ANK decomposition].

We set $dg = \frac{du}{u} \frac{dx\,dy}{y^2} d\theta$. on $GL_2(\mathbb{R}^+)$ and extend this to $GL_2(\mathbb{R})$].

(4) We can do something similar for $G = GL_n(\mathbb{A}_F)$ or $SL_n(\mathbb{A}_F)$. The archimedean measures become more complicated to specify explicitly, but

(4) We can do something similar for $G = GL_n(\mathbb{A}_F)$ or $SL_n(\mathbb{A}_F)$. The archimedean measures become more complicated to specify explicitly, but when $v \nmid \infty$ we will have $\mu(GL_n(\mathcal{O}_v)) = 1$.

§1.4 The Blichfeldt-Minkowski Lemma:

Lemma 1.4.1 (The Blichfeldt-Minkowski Lemma) Let F be a global field. There is a positive constant C $(= C(F))$ such that for any $x \in \mathbb{A}$ with $\|x\| > C$, there exists $\alpha \in F^*$ such that $\|\alpha\|_v \leq \|x_v\|_v$ $\forall$ $v \in M_F$.

<u>Proof</u>: We must have $|x|_v = 1$ for almost all v. Otherwise, there will be infinitely many finite places $v \in M_F$ such that $\|x\|_v < Np_v^{-1}$, where p_v is the prime at the place v. This would imply that $\|x\| = 0$. Let $\mu(D) = m$ (recall that D was the fundamental domain for the action of F on $\mathbb{A}$. Let m_1 be the measure of the set

$$E = \{ y \in \mathbb{A} \mid \|y_v\|_v \leq 1 \text{ if } v \nmid \infty \text{ and } \|y_v\|_v \leq \tfrac{1}{4} \text{ if } v \mid \infty \}$$

$$E = \{ y \in \mathbb{A} \mid \|y_v\|_v \le 1 \text{ if } v \nmid \infty \text{ and } \|y_v\|_v \le \tfrac{1}{4} \text{ if } v \mid \infty \}$$

Let $E_x = \{ y \in \mathbb{A} \mid \|y_v\|_v \le \|x_v\|_v \text{ if } v \nmid \infty \text{ and } \|y_v\|_v \le \|\frac{x_v}{4}\| \text{ if } v \mid \infty \}$

Then $\mu(E_x) = m_1 \|x\| > m_1 C$. Choose $C = \frac{m_0}{m_1}$, so $\mu(E_x) > m_0$. Since $\mu(E_x) > \mu(D)$, there must exist $\alpha_1, \alpha_2 \in E_x$ such that $\alpha_1 - \alpha_2 = \alpha \, (\neq 0) \in F$

Exercise: Check that $\alpha \in E_x$, i.e., $\|\alpha_v\|_v \le \|x_v\|_v$ for all $v \in M_F$.

Corollary 1.4.2: Let $\mathbb{I}^\rho = \{ x \in \mathbb{I} \mid \|x\| = \rho \}$. There exists a constant C_1 such that if $\rho > C_1$ and all $a \in \mathbb{I}^\rho$, there exists $\alpha \in F^*$ such that

$$1 \le \|\alpha_v a_v\|_v \le \rho \quad \forall \; v \in M_F.$$

Proof: Choose $\rho = C$ in Lemma 1.4.1. Then $\exists \; \alpha \in F^\times$ such that $\|\alpha_v^{-1}\|_v \le \|a_v\|_v \;\; \forall \; v \in M_F$. Hence

$$1 \le \|\alpha_v a_v\|_v \quad \forall \; v \in M_F.$$

We also have $\|\alpha_v a_v\|_v = \dfrac{\prod_w \|\alpha_w a_w\|}{\prod_{w \neq v} \|\alpha_w a_w\|} \le \dfrac{\rho}{1} = \rho.$

This proves the corollary.

Proof of Theorem 1.2.2. We note that $\mathbb{I}^\rho (\subseteq \mathbb{I})$ is

iPad 12:56 PM 83%
Maassforms

Proof of Theorem 1.2.2. We note that $\mathbb{I}^\rho (\subseteq \mathbb{I})$ is homeomorphic to $\mathbb{I}^1 (\subseteq \mathbb{I})$. Explicitly, we have

$$b \xrightarrow{\varphi} a_\rho b \quad ,$$

where $a_\rho = (\underbrace{\rho^{1/\alpha}, \rho^{1/\alpha}, \ldots, \rho^{1/\alpha}}_{S_\infty}, 1, \ldots 1, \ldots)$, is a homeomorphism from $\mathbb{I}^1$ to $\mathbb{I}^\rho$. This gives a homeomorphism at the level of quotients:

$$\mathbb{I}^1/F^* \left(\subseteq \mathbb{I}/F^*\right) \xrightarrow{\varphi} \mathbb{I}^\rho/F^* \left(\subseteq \mathbb{I}/F^*\right)$$

Thus it suffices to prove that C^ρ is compact.

As in the proof of Lemma 1.4.1. we can show that there is a finite set of places S such that $\|\alpha a\|_v = 1 \ \forall \ v \notin S$. (of course, $1 \leq \|\alpha a\|_v \leq \rho \ \forall \ v \in S$).

Define $A_{\rho,v} = \{x \in \mathbb{I} \mid \|x\|_w = 1$ if $w \neq v$, $1 \leq \|x\|_v \leq \rho\}$

Let $X = \prod_{v \in S} A_{\rho,v} \times \prod_{v \notin S} U_v$. Clearly X is compact.

Consider $\pi : \mathbb{I} \to C$, the natural quotient map. $\pi(X) \supseteq C^\rho$. Hence C^ρ is compact.

<u>Remarks</u> (1) If S is a set of finite places, $\mathbb{I}_S = \{x \in \mathbb{I} \mid \|x\|_v = 1 \ \forall \ v \notin S\}$ and

$F_S^* = \{\alpha \in F / \|\alpha\|_v = 1 \ \forall \ v \notin S\} = F^* \cap \mathbb{I}_S$

iPad 12:56 PM 83%
Maassforms

Remarks (1) If S is a set of finite places,

$\mathbb{I}_S = \{ x \in \mathbb{I} \mid \|x\|_v = 1 \ \forall\ v \notin S \}$ and

$F_S = \{ \alpha \in F \mid \|\alpha\|_v = 1 \ \forall\ v \notin S \}. = F^\times \cap \mathbb{I}_S$

Define $C_S = \mathbb{I}_S / F_S$ and $C_S^1 = \mathbb{I}_S^1 / F_S$.

Since $C_S^1 \subseteq C^1$, we see that C_S^1 is compact.

We have $\mathbb{I}/F^\times \mathbb{I}_S \simeq C/C_S$. When $S = S_\infty$

$\mathbb{I}/F^\times \mathbb{I}_{S_\infty}$ is (isomorphic to) the group of ideal classes which is finite. Hence C/C_{S_∞} is finite and so is C/C_S for all $S \supseteq S_\infty$.

(2) The topology on $\mathbb{I}^1$ is the same as the topology induced by the inclusion map $\mathbb{I}^1 \hookrightarrow \mathbb{A}$!

Exercise (ii) Given $S \supset S_\infty$, let $s = |S|$. We define

$\log : \mathbb{I}_S \longrightarrow \mathbb{R}^s$ by $(\ldots, a_v, \ldots) \to (\ldots, \log \|a\|_v, \ldots)_{v \in S}$

One checks easily that $\log(\mathbb{I}_S^1) \subseteq H$

$= \{ (z_1, \ldots, z_s) \in \mathbb{R}^s \mid z_1 + z_2 + \cdots + z_s = 1 \} \simeq \mathbb{R}^{s-1}$

Show that $\log(F_S)$ is a discrete subgroup of rank $(s-1)$ in H. Deduce Dirichlet's units theorem.

iPad 12:57 PM 82%
Maassforms

Exercise (ii): Show that $\mathbb{I}_{\mathbb{Q}} = \mathbb{Q}^{\times} \cdot \hat{\mathbb{Z}}^{\times} \cdot \mathbb{R}^{\times}$, where

$$\hat{\mathbb{Z}} = \varprojlim_{n} \mathbb{Z}/n\mathbb{Z} \simeq \prod_{p \nmid \infty} \mathbb{Z}_p .$$

§1.5 Strong Approximation.

Theorem 1.5.1 Let S be a finite set of places of F and let $w \in M_F$ with $w \notin S$. For each $v \in S$, suppose that we are given $a_v \in F_v$ and $\varepsilon_v \in \mathbb{R}^+$. There exists $x \in F$ such that

$$\|x_v - a_v\|_v \leq \varepsilon_v \quad \forall\, v \in S$$

$$\|x_v\|_v \leq 1 \quad \forall\, v \in M_F \setminus (\{w\} \cup S).$$

Proof: Let $\overline{D}$ be the closure of the fundamental domain D. Since $\overline{D}$ is compact, we see that there exist $t_v \in \mathbb{R}^+$, $v \mid \infty$ such that

$$\overline{D} \subseteq K = \{x \in \mathbb{A} \mid \|x_v\|_v \leq t_v \ \forall\, v \mid \infty \text{ and } \|t_v\|_v \leq 1 \text{ if } v \nmid \infty\}.$$

Let $t = \max\left(\left(\max_v t_v\right), 1\right)$

Note that for any $\beta \in F^*$, $\mathbb{A}_F = F + \beta D$

(If $a \in \mathbb{A}$ and $\beta \in F^*$, write $\beta^{-1} a = \gamma + b$ with $\gamma \in F$ and $b \in D$. Then $a = \beta\gamma + \beta b$ as desired).

iPad 1:03 PM 79%
Maassforms

Let $y \in \mathbb{A}$ be chosen so that

(i) $0 < \|y_v\|_v \le \varepsilon_v t_v^{-1} \ \forall\, v \in S \cap S_\infty$, $0 < \|y_v\|_v \le \varepsilon_v$, $v \in S - S_\infty$

(ii) $0 < \|y_v\|_v \le 1 \ \forall\, v \notin S \cup \{w\} \cup S_\infty$, $0 < \|y_v\|_v \le t_v^{-1} \ \forall$ $v \in (S \cup \{w\})^c \cap S_\infty$.

(iii) $\|y_w\|_w \ge C t \cdot \prod_{v \ne w} \|y_v\|_v^{-1}$

where C is the constant in Lemma 1.4.1.

Since $\|y\| \ge C$, $\exists\, \alpha \in F$ such that $\|\alpha_v\|_v \le \|y_v\|_v \ \forall\, v \in M_F$ (by Lemma 1.4.1)

Let $a = (\underbrace{a_v, a_v, \dots, a_v}_{\text{~~infinite~~ places of } S}, \underbrace{1, \dots 1, \dots}_{\text{~~finite~~ places. not in } S}) \in \mathbb{A}$

We can write $a = \gamma + \alpha d$ with $\gamma \in F$ and $d \in D$

$$\therefore \|(a-\gamma)_v\|_v \le \|\alpha_v d_v\|_v \le \begin{cases} \|y_v\|_v t_v \le \varepsilon_v & \text{if } v \in S \\ \|y_v\|_v \le 1 & \text{if } v \notin S \cup \{w\}. \end{cases}$$

This proves the theorem.

Theorem 1.5.2 (Strong approximation for $SL_2(\mathbb{A})$).

For any non-empty finite subset S of M_F, $SL_2(F)$ is dense in $SL_2(\mathbb{A}^S)$

Proof: Let $H = \overline{SL_2(F)}$ in $SL_2(\mathbb{A}^S)$. We will show that $H \supseteq SL_2(F_v)\ \forall\, v \notin S$. Now $SL_2(F)$

<u>Proof</u>: Let $H = \overline{SL_2(F)}$ in $SL_2(\mathbb{A}^S)$. We will show that $H \supseteq SL_2(F_v)$ $\forall$ $v \notin S$. Now $SL_2(F_v)$ is generated by $U^+(F_v) = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| x \in F_v \right\}$ and $U^-(F_v) = \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \middle| x \in F_v \right\}$, so it is enough to show that H contains U^+ and U^-. But $U^+, U^- \simeq G_a$ and $H \supseteq U^+(F)$ and $U^-(F)$. By Theorem 1.5.1 $H \supseteq U^+(F_v), U^-(F_v)$ and we are done.

Remark / Exercise: Use Theorem 1.5.2 to show that for any non-empty finite subset S of M_F, $SL_n(F)$ is dense in $SL_n(\mathbb{A}^S)$. The same ideas work for any connected semisimple F-group G that is simply connected and split.

Corollary 1.5.3: $SL_2(F) \cdot \prod_{v \notin S} SL_2(\mathcal{O}_v) = SL_2(\mathbb{A}^S)$. Using the exercise we have, more generally,

$$SL_n(F) \cdot \prod_{v \notin S} SL_n(\mathcal{O}_v) = SL_n(\mathbb{A}^S).$$

The result holds for any open subgroup of $\prod_{v \notin S} SL_n(\mathcal{O}_v)$.

We now specialise to the case $F = \mathbb{Q}$

Theorem 1.5.4 (Strong approximation for SL_2)

iPad 1:03 PM 78%
Maassforms

We now specialise to the case $F = \mathbb{Q}$

Theorem 1.5.4 (Strong approximation for SL_n)

Let U be a subgroup of finite index in $SL_n(\hat{\mathbb{Z}}) = \prod_{p \nmid \infty} SL_n(\mathbb{Z}_p)$. Then

$$SL_n(\mathbb{A}) = SL_n(\mathbb{Q}) \cdot U \cdot SL_n(\mathbb{R})$$

Proof: Take $S = S_\infty$ in Corollary 1.5.3.

If we use Exercise (ii) of §1.4 we obtain

Theorem 1.5.5 (Strong approximation for GL_n)

Let U be a subgroup of finite index in $GL_n(\hat{\mathbb{Z}})$ such that $\det(U) = \hat{\mathbb{Z}}^\times$. Then

$$GL_n(\mathbb{A}) = GL_n(\mathbb{Q}) \cdot U \cdot GL_n(\mathbb{R})^+.$$

$$(GL_n(\mathbb{R}^+) = \{ g \in GL_n(\mathbb{R}) \mid \det g > 0 \}.$$

Corollary 1.5.6: With notation as above, let $\Gamma := SL_2(\mathbb{Z}) \cap U$.

Then

$$\Gamma \backslash GL_2(\mathbb{R}^+) \xrightarrow{\cong} GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) / U \qquad 1.5.6\ (i)$$

$$\Gamma \backslash \mathbb{H} \xrightarrow{\cong} GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) / U \cdot U_\infty \qquad 1.5.6\ (ii)$$

iPad 1:03 PM 78%
Maassforms

where $U_\infty = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \Big| \ a, b \in \mathbb{R} \right\}$.

[Notice that $U_\infty \simeq \mathbb{R}^+ \times SO_2 \simeq \mathbb{C}^*$].

The most important examples of U will be the following:

(1) $K_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Big| \ c \equiv 0 \pmod N \right\} \subseteq GL_2(\hat{\mathbb{Z}})$

(2) $K_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Big| \ c \equiv 0, d \equiv 1 \pmod N \right\} \subseteq GL_2(\hat{\mathbb{Z}})$.

We have $K_0(N) = \prod_{v \nmid \infty} K_{0,v}(m_v)$, where $N = \prod_{v \nmid \infty} p_v^{m_v}$.

Similarly $K_1(N) = \prod_{v \nmid \infty} K_{1,v}(m_v)$

[$K_{0,v}(m_v) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}_{p_v}) \Big/ c \equiv 0 \ (\mathrm{mod}\ p_v^{m_v}) \right\}$].

Note that $K(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Big| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N \right\}$

does not satisfy the condition $\det(K(N)) = \mathbb{Z}^\times$.

As a variant on 1.5.6 (i) and (ii) we also have

$$\Gamma \backslash \mathbb{H} \simeq GL_2(\mathbb{Q}) Z(\mathbb{A}) \backslash GL_2(\mathbb{A}) / K_\infty U \qquad 1.5.6 \text{ (iii)}$$

where $K_\infty = SO_2(\mathbb{R})$; when $U = GL_n(\hat{\mathbb{Z}})$, we get

$$SL_2(\mathbb{Z}) \backslash \mathbb{H} \simeq GL_2(\mathbb{Q}) Z(\mathbb{A}) \backslash GL_2(\mathbb{A}) / K \qquad 1.5.6 \text{ (iv)}$$

where $K_\infty = SO_2(\mathbb{R})$; when $U = GL_n(\hat{\mathbb{Z}})$, we get

$$SL_2(\mathbb{Z})\backslash\mathbb{H} = GL_2(\mathbb{Q})\cdot Z(\mathbb{A})\backslash GL_2(\mathbb{A})/K, \quad 1.5.6\ (iv)$$

where $K = K_\infty \cdot K_f$; $K_\infty = SO_2(\mathbb{R})$,

$$K_f = \prod_{p \nmid \infty} GL_n(\mathbb{Z}_p) \cong GL_n(\hat{\mathbb{Z}}).$$

When $U = K_0(N)$, we have

$$\Gamma_0(N)\backslash\mathbb{H} \simeq GL_2(\mathbb{Q}) Z(\mathbb{A})\backslash GL_2(\mathbb{A})/K_\infty \cdot K_0(N) \quad 1.5.6\ (v)$$

and if $U = K_1(N)$,

$$\Gamma_1(N)\backslash\mathbb{H} \simeq GL_2(\mathbb{Q}) Z(\mathbb{A})\backslash GL_2(\mathbb{A})/K_\infty K_1(N)$$

1.5.6 (vi)

Exercise 1.5.(i): Show that strong approximation implies that the natural quotient map

$$SL_2(\mathbb{Z}) \longrightarrow SL_2(\mathbb{Z}/N\mathbb{Z})$$

is a surjection for all N.

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \pmod N \right\}.$$

Lecture 3. : Part 1.

Recall that the k-Laplacian

$$\Delta_k = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\frac{\partial}{\partial x} \quad \text{on } \mathcal{C}^\infty(\mathbb{H})$$

commutes with the action of $G = GL_2(\mathbb{R})^+$:

$$(\Delta_k f)|_k g = \Delta_k(f|_k g)$$

Let Γ be a discontinuous subgroup of G. (e.g. $\Gamma = \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;/\; c \equiv 0 \pmod N \right\}$)

We will assume $\Gamma \leq G_1 = SL_2(\mathbb{R})$ and $-I \in \Gamma$.

Let $\chi : \Gamma \longrightarrow S^1$ and $\omega : Z(\mathbb{R}) \longrightarrow S^1$ be (unitary) characters such that $\chi(-I) = \omega(-1)$.

[$Z(\mathbb{R})$ = centre of $GL_2(\mathbb{R})^+ \simeq \mathbb{R}^*$].

We define $\mathcal{C}^\infty(\Gamma\backslash\mathbb{H}, \chi, k)$ as the functions $f \in \mathcal{C}^\infty(\mathbb{H})$ such that $f|_k \gamma = \chi(\gamma) f$.

Cusps: $SL_2(\mathbb{C})$ acts transitively on $\mathbb{C} \cup \{\infty\}$ (the Riemann sphere) by fractional linear transformations.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \begin{cases} \dfrac{az+b}{cz+d}, & \text{if } z \neq \infty \\ a/c & \text{if } z = \infty,\ c \neq 0 \\ \infty & \text{if } z = \infty,\ c = 0 \end{cases}$$

Notice that $SL_2(\mathbb{R})$ can be characterised as the subgroup $\{ h \in SL_2(\mathbb{C}) \;/\; h(\mathbb{H}) \subseteq \mathbb{H} \}$.

$SL_2(\mathbb{R})$ acts on $\mathbb{R} \cup \{\infty\}$.

[Remark: We can think of $\mathbb{C} \cup \{\infty\}$ as $\mathbb{P}^1(\mathbb{C})$, the set of lines in $\mathbb{C}^2$. Then $SL_2(\mathbb{C})$ obviously acts on $\mathbb{C}^2$ and hence on $\mathbb{P}^1(\mathbb{C})$. Similarly $SL_2(\mathbb{R})$ acts on $\mathbb{R} \cup \{\infty\}$ $(\simeq \mathbb{P}^1(\mathbb{R}))$.

Definition: An element $\gamma \in SL_2(\mathbb{R})$ is called parabolic if $|\mathrm{tr}(\gamma)| = 2$.

Definition: A point "a" $\in \mathbb{R} \cup \{\infty\}$ is called a cusp of Γ if Γ contains a parabolic element γ_0 such that $\gamma_0 \cdot a = a$.

Notice that if γ is any element of Γ and "a" is a cusp of Γ, then $\gamma \cdot a$ is also a cusp of Γ since $(\gamma \gamma_0 \gamma^{-1}) \cdot \gamma \cdot a = a$. Thus the word cusp is often used to refer to the whole orbit Γa rather than the point "a".

Example 1: $\Gamma = SL_2(\mathbb{Z})$; $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \infty$, so "$\infty$" is a cusp for Γ. In fact $\Gamma \cdot \infty = \mathbb{Q}$ is the only cusp.

Example 2: $\Gamma = \Gamma(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv I \pmod 2 \right\}$

Then $0, 1, \infty$ are cusps ; $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \cdot \infty = \infty$ $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \cdot 0 = 0$.

Find γ parabolic in $\Gamma(2)$ s.t. $\gamma \cdot 1 = 1$.

§ 2.0 Maass forms:

We first define the Maass differential operators on $\mathcal{C}^\infty(H)$:

$$R_k : iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{k}{2} = (z-\bar{z})\frac{\partial}{\partial z} + \frac{k}{2} \qquad (2.1)$$

$$L_k : -iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{k}{2} = -(z-\bar{z})\frac{\partial}{\partial \bar{z}} - \frac{k}{2} \qquad (2.2)$$

and

$$\Delta_k = -L_{k+2}R_k - \frac{k}{2}\left(1+\frac{k}{2}\right) = -R_{k-2}L_k + \frac{k}{2}\left(1-\frac{k}{2}\right). \qquad (2.3)$$

Δ_k is called the weight k non-Euclidean Laplacian

It can be written explicitly as

$$\Delta_k = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\frac{\partial}{\partial x} \qquad (2.4)$$

For each k we define a right action of $G = GL_2(\mathbb{R}^+)$ on $\mathcal{C}^\infty(\mathbb{H})$ as follows. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$.

$$f|_k g = \left(\frac{c\bar{z}+d}{|cz+d|}\right)^k f\left(\frac{az+b}{cz+d}\right) \qquad (2.5)$$

<u>Lemma 2.1</u>: The k-Laplacian commutes with the G action described by (2.5):

$$(\Delta_k f)|_k g = \Delta_k(f|_k g) \qquad (2.6)$$

Proof: The proof will require the following exercise:

Exercise 2.2: Show that for $f \in \mathcal{C}^\infty(\mathbb{H})$

$$(R_k f)|_{k+2}\, g = R_k(f|_k g) \qquad (2.7)$$

$$(L_k f)|_{k-2}\, g = L_k(f|_k g) \qquad (2.8)$$

Use (2.7) and (2.8) to deduce (2.6)

Let Γ be a discontinuous subgroup of G. We will assume that $-I \in \Gamma$ (otherwise simply work with the subgroup generated by $-I$ and Γ). We will also assume without loss of generality that $\Gamma \subseteq SL_2(\mathbb{R})$.

The group $GL_2^+(\mathbb{R})$ comes equipped with a Haar measure (a (left)-invariant outer regular Borel measure - in fact $G = GL_2(\mathbb{R}^+)$ is unimodular, that is, the measure is both left and right invariant). Recall that the Iwasawa decomposition for G states that every element $g \in G$ can be written in the form

$$g = \begin{pmatrix} z & \\ & z \end{pmatrix} \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix} K_\theta \ , \quad K_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$x, y, z, \theta \in \mathbb{R}, \quad y, z > 0. \qquad (2.9)$$

We take the Haar measure on $SO_2 = \{ K_\theta \mid \theta \in \mathbb{R} \}$ to be $d\theta$. A Haar measure on G is given by

$$dg = \frac{dz}{z} \frac{dx\, dy}{y^2} d\theta \qquad (2.10)$$

If we set $G^1 = SL_2(\mathbb{R})$, $\frac{dx\,dy}{y^2} d\theta$ is a G^1 invariant measure on G^1. We also obtain a G-invariant (and hence, also G^1-invariant) measure $\frac{dx\,dy}{y^2}$ on $\mathbb{H}$. This descends to a G-invariant measure on $\Gamma \backslash \mathbb{H}$. We will assume that

$\mathrm{vol}(\Gamma\backslash\mathbb{H}) < \infty$.

The primary examples of Γ that we have in mind are

$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}$ and

$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0,\ d \equiv 1 \pmod{N} \right\}$.

We will also sometimes need the subgroup

$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$.

Let

Lecture 3. : Part 1.

Recall that the k-Laplacian

$$\Delta_k = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\frac{\partial}{\partial x} \quad \text{on } \mathcal{C}^\infty(\mathbb{H})$$

commutes with the action of $G = GL_2(\mathbb{R})^+$:

$$(\Delta_k f)|_k g = \Delta_k (f|_k g)$$

Let Γ be a discontinuous subgroup of G. (e.g.

$\Gamma = \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0 \pmod N \right\}$)

We will assume $\Gamma \leq G_1 = SL_2(\mathbb{R})$ and $-I \in \Gamma$.

Let $\chi : \Gamma \longrightarrow S^1$ and $\omega : Z(\mathbb{R}) \longrightarrow S^1$ be

(unitary) characters such that $\chi(-I) = \omega(-1)$.

[$Z(\mathbb{R})$ = centre of $GL_2(\mathbb{R})^+ \simeq \mathbb{R}^*$].

We define $\mathcal{C}^\infty(\Gamma \backslash \mathbb{H}, \chi, k)$ as the functions

$f \in \mathcal{C}^\infty(\mathbb{H})$ such that $f|_k \gamma = \chi(\gamma) f$.

Cusps: $SL_2(\mathbb{C})$ acts transitively on $\mathbb{C} \cup \{\infty\}$ (the Riemann sphere) by fractional linear transformations.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \begin{cases} \dfrac{az+b}{cz+d}, & \text{if } z \neq \infty \\ a/c & \text{if } z = \infty,\ c \neq 0 \\ \infty & \text{if } z = \infty,\ c = 0 \end{cases}$$

In $SL_2(\mathbb{C})$, we may view $SL_2(\mathbb{R})$ as $\{ h \in SL_2(\mathbb{C}) \mid h(\mathbb{H}) \subseteq \mathbb{H} \}$. It acts on $\mathbb{R} \cup \{\infty\}$.

Remark: We may view the Riemann sphere $\mathbb{C} \cup \{\infty\}$ as the set of lines in $\mathbb{C}^2$. Then $SL_2(\mathbb{C})$ obviously acts on $\mathbb{C}^2$ and hence on $\mathbb{P}^1(\mathbb{C})$. Similarly $SL_2(\mathbb{R})$ acts on $\mathbb{R} \cup \{\infty\}$ $(\simeq \mathbb{P}^1(\mathbb{R}))$.

Definition: An element $\gamma \in SL_2(\mathbb{R})$ is called parabolic if $|\mathrm{tr}(\gamma)| = 2$.

Definition: A point "a" $\in \mathbb{R} \cup \{\infty\}$ is called a cusp of Γ if Γ contains a parabolic element γ_0 such that $\gamma_0 \cdot a = a$.

Notice that if γ is any element of Γ and "a" is a cusp of Γ, then $\gamma \cdot a$ is also a cusp of Γ since $(\gamma \gamma_0 \gamma^{-1}) \cdot \gamma \cdot a = a$. Thus the word cusp is often used to refer to the whole orbit Γa rather than the point "a".

Example 1: $\Gamma = SL_2(\mathbb{Z})$; $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \infty$, so "$\infty$" is a cusp for Γ. In fact $\Gamma \cdot \infty = \mathbb{Q}$ is the only cusp.

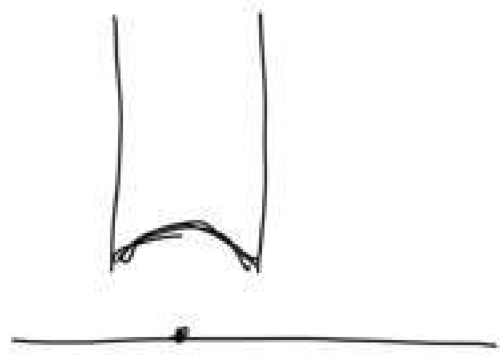
Example 2: $\Gamma = \Gamma(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv I \pmod 2 \right\}$

Then $0, 1, \infty$ are cusps ; $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \cdot \infty = \infty$ $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \cdot 0 = 0$.

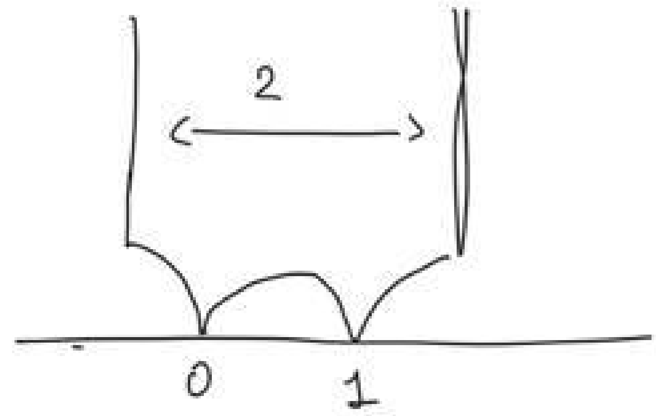
Find γ parabolic in $\Gamma(2)$ s.t. $\gamma \cdot 1 = 1$.

Remarks: 1) Again, the points $0, 1$ and ∞ are the only cusps (upto $\Gamma(2)$-equivalence).

2. Geometrically, the cusps are the points where the fundamental domain "touches" $\mathbb{R} \cup \{\infty\}$.



$\Gamma = SL_2(\mathbb{Z})$



$\Gamma = \Gamma(2)$

The pictures above give the fundamental domains for the groups $SL_2(\mathbb{Z})$ and $\Gamma(2)$,

Lecture 3, Part 2: If $f \in \mathcal{C}^\infty(\Gamma \backslash \mathbb{H}, \chi, k)$ and infinity is a cusp of Γ, we will say that

(1) f has moderate growth at ∞ if there are constants $C > 0$, $l \in \mathbb{N}$ such that

$$|f(x+iy)| \leq C y^l \quad \text{if} \quad y \geq 1 \qquad (2.9)$$

(2) f has rapid decay if for each $\alpha > 0$, $\exists$ $C_\alpha > 0$ s.t $|f(x+iy)| \leq C_\alpha y^{-\alpha}$ (2.10)

(3) f is cuspidal at ∞ if

$$\int_0^t f(z+u)\, du = 0 \quad \forall z, \qquad (2.11)$$

where $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \in \Gamma$. Such an element exists since we have assumed that "∞" is a cusp of Γ.

If "a" is a cusp of Γ and $h \cdot \infty = a$, then "∞" is a cusp of $\tilde{\Gamma} = h^{-1} \Gamma h$. Then, $\tilde{f} = f|_k h \in \mathcal{C}^\infty(\tilde{\Gamma} \backslash \mathbb{H}, \tilde{\chi}, k)$, where $\tilde{\chi}(\gamma) = \chi(\gamma_1)$, $\gamma_1 = h \gamma h^{-1}$ is a character of $\tilde{\Gamma}$. We will say that f has moderate growth / has rapid decay / is cuspidal at "a" if $\tilde{f}$ has moderate growth / has rapid decay / is cuspidal at "∞".

A Maass form of weight is an element of $\mathcal{C}^\infty(\Gamma \backslash \mathbb{H}, \chi, k)$ which is an eigenfunction of Δ_k, satisfies $f(g k_\theta) = e^{2\pi i k} f(g)$* for all $g \in G$, and $k_\theta \in SO_2(\mathbb{R})$ and which is of moderate growth at all the cusps of Γ.

Exercise 2.3: If f satisfies (2.11), then f satisfies (2.10). If satisfies (2.10), then f satisfies (2.09).

Exercise 2.4: Show that $y^{k/2} f(z)$ is a Maass form of weight k and eigenvalue $\frac{k}{2}\left(1-\frac{k}{2}\right)$ of Δ_k, where $f(z)$ is a holomorphic modular form of weight k.

Do non-holomorphic Maass forms exist?

Weyl's Law for $SL_2(\mathbb{R})$: Selberg.

In much greater generality: Lindenstrauss - Venkatesh

(see Goldfeld: Automorphic forms on $GL_n(\mathbb{R})$)

* During the lecture I was not sure whether $e^{2\pi i k}$ was the correct constant so I called the constant "μ_k". As someone in the lecture remarked, when "k" is an integer this factor becomes identically 1, so we get SO_2-invariant functions in this case.

If $f, g \in \mathcal{C}^\infty(\Gamma \backslash \mathbb{H}, \chi, k)$, $f(z)\overline{g(z)}$ is Γ-invariant. Hence,

$$\langle f, g \rangle = \int_{\Gamma \backslash \mathbb{H}} f(z)\overline{g(z)}\, dz$$

defines an inner product on this space. We denote by $L^2(\Gamma \backslash \mathbb{H}, \chi, k)$ the completion of $\mathcal{C}^\infty(\Gamma \backslash \mathbb{H}, \chi, k)$ with respect to this inner product.

Lecture 3, Part 3:

§ 3.1: Classical Automorphic forms

Recall that $\chi : \Gamma \longrightarrow S^1$ and $\omega : Z(\mathbb{R}) \longrightarrow S^1$ are (unitary characters such that $\chi(-I) = \omega(-1)$ ($Z(\mathbb{R}) \simeq \mathbb{R}^*$ is the centre of $G = GL_2(\mathbb{R})^+$).

We define the space $\mathcal{C}^\infty(\Gamma \backslash G, \chi, \omega)$ as

$$\{ F : G \longrightarrow \mathbb{C} \mid \text{s.t. } F(\gamma g) = \chi(\gamma) F(g),\ \gamma \in \Gamma,\ g \in G. \text{ and } F(zg) = \omega(z) F(g),\ z \in Z(\mathbb{R})\ g \in G \}.$$

Since $G \subseteq M_2(\mathbb{R})$, we may view $v_g = (g, \det g^{-1})$ as a vector in $M_2(\mathbb{R}) \oplus \mathbb{R}^5$. Let $\ell(v_g)$ denote the length of the vector v_g and define $\| g \| = \ell(v_g)$. (note that we could also use the L^1-norm on $\mathbb{R}^5$).

Assume that "∞" is a cusp of Γ. We will say that $f \in \mathcal{C}^\infty(\Gamma \backslash G, \chi, \omega)$

1) has moderate growth at ∞ if there exist $C, K > 0$ such that $|f(g)| \leq C \| g \|^K$ (3.1)

2) has rapid decay at ∞ if for every $N > 0$, there exists $C_N > 0$ such that $|f(g)| \leq C_N \| g \|^{-N}$ (3.2)

3) is cuspidal at ∞ if

$$\int_0^t F\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) dx = 0 \qquad (3.3)$$

[Since ∞ is a cusp of Γ, there is a $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \in \Gamma$].

We define the space of classical Automorphic forms $\mathcal{A}(\Gamma \backslash G, \chi, \omega)$ as the space of functions $F(g) \in \mathcal{C}^\infty(\Gamma \backslash G, \chi, \omega)$ such that

1) F is an eigenfunction of the operator

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + y \frac{\partial^2}{\partial x \partial \theta}$$

2) $F(g k_\theta) = e^{2\pi i k} F(g) \quad g \in G, \; k_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

for some $k \in \mathbb{R}$.

3) F is of moderate growth at all the cusps of Γ*

* Note that we have defined the notions of moderate growth, rapid decay and cuspidality only for the cusp at ∞. If $a \neq \infty$ is a cusp of Γ and $h.a = \infty$, we can define these conditions for the function F at "a" in terms of those for $\tilde{F}$ at ∞ just as we did in the case of Maass forms.

The space of functions $F \in \mathcal{A}(\Gamma \backslash G, \chi, \omega)$ satisfying (3.3) is denoted $\mathcal{A}_0(\Gamma \backslash G, \chi, \omega)$ and is called the space of cusp forms.

We also have the space $L^2(\Gamma \backslash G, \chi, \omega)$.

These will be L^2-functions $F(g)$ such that $F(\gamma g) = \chi(\gamma) F(g)$ and $F(z g) = \omega(z) F(g)$ as before. The subspace $L^2_0(\Gamma \backslash G, \chi, \omega)$ is defined by the additional condition (3.3) but this will now be a condition only for "g" almost everywhere.

The group G acts on functions by right translations. We will call this representation on $L^2(\Gamma\backslash G, \chi, \omega)$ "ρ".

Fact: $L^2_0(\Gamma\backslash G, \chi, \omega)$ is a ρ-stable subspace

Theorem 3.1.1:

$$L^2_0(\Gamma\backslash G, \chi, \omega) = \bigoplus_{\pi} \mathcal{H}_\pi ,$$

where π runs over a (countable) set of irreducible representations of G (which are ρ-stable). Each equivalence class of irreducible representations occurs with at most finite multiplicity.

RR6

RR7

Lecture 4, Part 1:

§ 3.2: From Maass forms to Classical automorphic forms.

The basic point of this section is that Maass forms give rise to automorphic forms.

Let f be a Maass form of weight k for Γ with associated character χ (sometimes this character is called the Nebentypus of f).

Let $F(g) = (f|_k g)(i)$.

Proposition 3.2.1: The function F lies in $\mathcal{A}(\Gamma\backslash G, \chi, \omega)$ where ω is the character on $Z(\mathbb{R}) \simeq \mathbb{R}^*$ that is trivial on $\mathbb{R}^+$ and which agrees with χ on $(-I)$. Further, if f is a Maass cusp form, F will be a cusp form on G.

Note that it is not surprising that we can transfer functions from $\mathbb{H}$ to G in this way. After all $\mathbb{H} \simeq G/_{Z(\mathbb{R})SO_2(\mathbb{R})}$ and the group $SO_2(\mathbb{R})$ is exactly the stabiliser of "i". for the action of G on $\mathbb{H}$.

The fact that F is $SO_2(\mathbb{R})$ $(=K_\infty)$-stable is immediate from the corresponding property for the Maass form f. Indeed $F(gk_\theta) = e^{2\pi i k} F(g)$.
Similarly, the fact that $\Delta F = \lambda F$ is true because f is an eigenfunction of Δ_k on $\mathcal{C}^\infty(\mathbb{H})$.

§4: Adelising the classical automorphic forms:

We will realise the classical automorphic forms on G as functions on the adèle group $GL_2(\mathbb{A}_\mathbb{Q})$.
We will carry this out when $\Gamma = \Gamma_0(N)$.
In this case the associated character will be a Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z})^* \longrightarrow \mathbb{C}$

Let $U(N) = \mathbb{R}^+ \times \prod\limits_{\substack{v \mid N \\ v \nmid \infty}} U_v(N) \times \prod\limits_{\substack{v \nmid N \\ v \nmid \infty}} U_v$ and

$$V(N) = \mathbb{R}^+ \times \prod_{\substack{v \mid N \\ v \nmid \infty}} U_v(N) \times \prod_{\substack{v \nmid N \\ v \nmid \infty}}{}' \mathbb{Q}_p.$$

Then $\mathbb{I}/\mathbb{Q}^* \simeq \mathbb{Q}^* V(N)/\mathbb{Q}^* \simeq V(N)/\mathbb{Q}^* \cap V(N)$.

We have $V(N)/\mathbb{Q}^* \cap V(N) \xrightarrow{\text{proj}} V(N)/U(N)(\mathbb{Q}^* \cap V(N)) \simeq (\mathbb{Z}/N\mathbb{Z})^*$.
Thus any Dirichlet character χ can be viewed as an idèle class character $\tilde{\chi} : \mathbb{I}/\mathbb{Q}^* \to S^1$.

Lecture 4, Part 2:

Let f be a Maass form of weight k for $\Gamma_0(N)$ and let $\chi : (\mathbb{Z}/N\mathbb{Z})^* \to S^1$ be the associated character. Let $F(g) = (f|_k g)(i)$ be the associated classical automorphic form.

Let $\omega : \mathbb{I}/\mathbb{Q}^* \to S^1$ be the adelisation of χ.

One checks that

$$\chi(d) = \prod_{v|N} \omega_v^{-1}(d_v) \quad \text{for } d \in \mathbb{Q}^*.$$

We define a character $\lambda : K_0(N) \to S^1$ by

$$\lambda\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \prod_{v|N} \omega_v(d_v) \qquad (4.1)$$

By strong approximation for SL_2, we know that $g = \gamma g_\infty k_0$ where $\gamma \in GL_2(\mathbb{Q})$, $g_\infty \in G$ and $k_0 \in K_0(N)$.

Define : $\quad \phi(g) = \lambda(k_0) F(g_\infty)$

Exercise 4.1 : Show that $\phi(g)$ is well-defined.

Exercise 4.2 : Show that $\phi\left(\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} g\right) = \omega(z)\phi(g)$ $\forall z \in \mathbb{I}$ and $g \in G$.

ϕ is an automorphic form on $GL_2(\mathbb{A}_\mathbb{Q})$ with central character ω, that is,

$\phi \in \mathcal{C}^\infty(GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}), \omega)$

$= \{ f \in \mathcal{C}^\infty(GL_2(\mathbb{A})) \mid f(zg) = \omega(z) f(g) \}$,

$\Delta \varphi_\infty = \lambda \varphi_\infty$ for some λ, (4.1)

ϕ is $K (= K_\infty \cdot K_f)$ - finite and (4.2)

$\exists N$ such that $|\phi(g)| \leq C \|g\|^N$ for some constant $C > 0$. (4.3)

We define some of the terms above:

ϕ is K-finite means that the translates of ϕ by K span a finite dimensional vector space

The norm $\|g\|$ is defined by first defining $\|g_v\|_v = \max_i |x_i|_v$ where the x_i are coordinates of $(g_v, \det g_v^{-1}) \in M_2(\mathbb{R}) \oplus \mathbb{R} \simeq \mathbb{R}^5$.

Now set $\|g\| = \prod_v \|g_v\|_v$. (one checks easily that $\|g_v\|_v = 1$ $\forall' v$). Now the moderate growth condition makes sense for $GL_2(\mathbb{A}_\mathbb{Q})$.

As you can see most of what we have said makes sense for GL_n. The condition (4.1) will need some revision, though. When the group is GL_2, the operators Δ and I generate "the centre Z of the universal enveloping algebra of G".

For GL_n, $n > 2$, Z will once again be a family of commuting partial differential operators but will require more generators. Thus (4.1) is replaced by the requirement that ϕ_∞ be Z-finite. The conditions (4.2) and (4.3) remain the same for automorphic forms on GL_n.

References: Lectures 3 and 4 are largely from D. Bump "Automorphic forms and Representations", Sections 2.1, 3.2, 3.3 and 3.6. Lectures 1 and 2 have more varied sources: Lang's "Algebraic Number Theory", and MIT notes on Algebraic Number Theory 2015 (Lecture 22) as well as Brian Conrad's note on

RR8 Strong approximation.