Chow groups, proper push forwards, flat pullbacks

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Abstract

These notes are supplementary reading for my introductory series of six talks in the NCM workshop on Intersection Theory at IIT Bombay in May 2022. They include details for which there is not enough time in the lectures. The notes are expository, expanding on Fulton's 'Intersection Theory' [F] Chapter 1 and Appendix A, except for the following two topics. The account of meromorphic functions expands on that in Mumford's 'Curves on an algebraic surface' [M]. The proof that flat pullbacks preserve rational equivalence is taken from [N].

Contents

1 Regular functions and meromorphic functions

Regular functions

1.1 Exercise. A presheaf $\mathcal{A}$ of rings on a topological space X is a sheaf if and only if for any open $U \subset X$ and an open cover $(U_i \to U)_{i\in I}$ of U, the following diagram of rings is exact (means the first homomorphism is injective, and its image is exactly the subset where the two homomorphisms on the right coincide), where $U_{jk} = U_j \cap U_k$.

$$\mathcal{A}(U) \to \prod_{i\in I} \mathcal{A}(U_i) \rightrightarrows \prod_{(j,k)\in I\times I} \mathcal{A}(U_{jk}).$$

The various ring homomorphisms in the above diagram are defined to be as follows. The ring homomorphism $\mathcal{A}(U) \to \prod_{i\in I} \mathcal{A}(U_i)$ is induced by the restrictions $\mathcal{A}(U) \to \mathcal{A}(U_i)$. The top homomorphism in the pair of homomorphisms $\prod_{i\in I} \mathcal{A}(U_i) \rightrightarrows \prod_{(j,k)\in I\times I} \mathcal{A}(U_{jk})$ is induced by the homomorphisms $\prod_{i\in I} \mathcal{A}(U_i) \rightrightarrows \mathcal{A}(U_{jk}) : (f_i) \mapsto f_j|_{U_{jk}}$ and the bottom homomorphism in $\prod_{i\in I} \mathcal{A}(U_i) \rightrightarrows \prod_{(j,k)\in I\times I} \mathcal{A}(U_{jk})$ is induced by the homomorphisms $\prod_{i\in I} \mathcal{A}(U_i) \rightrightarrows \mathcal{A}(U_{jk}) : (f_i) \mapsto f_k|_{U_{jk}}$.

A similar statement also applies to sheaves of sets or groups etc.

1.2 Defining the sheaf $\mathcal{A}$ using a special basis of open sets for X. Let X be a topological space and let B be a basis for the open sets of X which enjoys the following special property: if $U, V \in B$ then $U \cap V \in B$. Suppose we are given a commutative ring $\mathcal{B}(U)$ for each $U \in B$ and a ring homomorphism $s^U_V : \mathcal{B}(U) \to \mathcal{B}(V)$ for each $V \subset U$ with $U, V \in B$ such that the following properties hold:
(1) $s^U_U = \mathrm{id} : \mathcal{B}(U) \to \mathcal{B}(U)$.
(2) If $U, V, W \in B$ with $U \supset V \supset W$ then $s^U_W = s^V_W \circ s^U_V : \mathcal{B}(U) \to \mathcal{B}(W)$.
(3) If $U \in B$ and if $U = \cup_{i\in I} U_i$ where each $U_i \in B$ and $U_i \neq U_j$ for $i \neq j$, then the following diagram is exact in the category *Rings* of all commutative rings:

$$\mathcal{B}(U) \to \prod_{i\in I} \mathcal{B}(U_i) \rightrightarrows \prod_{(j,k)\in I\times I} \mathcal{B}(U_{jk})$$

where $U_{jk} = U_j \cap U_k$, which again belongs to B by assumption. Here, the definitions of the arrows are analogous to those in Exercise 1.1, using the given homomorphisms s^U_V in place of the restriction homomorphisms r^U_V. Then there exists a sheaf of rings $\mathcal{A}$ on X (see below for a construction), and for each $U \in B$ a ring isomorphism $\theta_U : \mathcal{B}(U) \to \mathcal{A}(U)$, such that if $U, V \in B$ with $U \supset V$, then $\theta_V \circ s^U_V = r^U_V \circ \theta_U$ where r^U_V are the restriction homomorphisms for $\mathcal{A}$. Moreover, such a pair $(\mathcal{A}, (\theta_U)_{U\in B})$ is unique up to a unique isomorphism.

Construction of $\mathcal{A}$. For an arbitrary open subset $V \subset X$, define the ring

$$\mathcal{A}(V) = \lim_{\{U \,|\, U\in B,\, U\subset V\}} \mathcal{B}(U)$$

where the limit (means inverse limit) is taken in the category of commutative rings. If $V \subset W$ are opens in X, then define the restriction map $r^W_V : \mathcal{A}(W) \to \mathcal{A}(V)$ as follows. For any $U \in B$, if $U \subset V$ then $U \subset W$, hence there is natural ring homomorphism $r_U : \lim_{\{U' \,|\, U'\in B,\, U'\subset W\}} \mathcal{B}(U') \to \mathcal{B}(U)$. These homomorphisms r_U together induce a natural ring homomorphism

$$r^W_V : \mathcal{A}(W) = \lim_{\{U' \,|\, U'\in B,\, U'\subset W\}} \mathcal{B}(U') \to \lim_{\{U \,|\, U\in B,\, U\subset V\}} \mathcal{B}(U) = \mathcal{A}(V).$$

Exercise. Show that the above construction indeed works: first show that $(\mathcal{A}(U), r^W_V)$ is a presheaf, then check it is a sheaf, and show that there exists a family of isomorphisms $(\theta_U : \mathcal{B}(U) \to \mathcal{A}(U))_{U\in B}$ with the required properties.

1.3 Definition of an affine scheme. Let A be a ring. To define the scheme $\mathrm{Spec}\, A$, we begin with its underlying set $|\mathrm{Spec}\, A|$ consisting of all prime ideals $\mathfrak{p} \subset A$ and give it the Zariski topology with basic open subsets $U_f = \{\mathfrak{p} \,|\, f \notin \mathfrak{p}\}$ where $f \in A$. This is a special basis in the sense of statement 1.2 above, as $U_f \cap U_g = U_{fg}$. The structure sheaf $\mathcal{O}_A$ on the topological space $|\mathrm{Spec}\, A|$ is defined by taking $\mathcal{O}_A(U_f) = A_f$ for the basic open sets, and for $U_g \subset U_f$ taking the restriction map $\mathcal{O}_A(U_f) \to \mathcal{O}_A(U_g)$ to be the localization homomorphism $A_f \to A_g = (A_f)_g$ (which inverts g) : see the two exercises below which shows why this works.

Exercise: (i) If $f, g \in A$ then show that that following are equivalent: (a) $U_g \subset U_f$, (b) the image of f under the localization homomorphism $\phi_{A,g} : A \to A_g$ is invertible in A_g, (c) there exists an element $x \in A$ and an integer $n \geq 0$ such that $xf = g^n$.
(ii) Deduce that if $f, g \in A$ such that $U_g \subset U_f$ then there exists a unique A-algebra homomorphism $\phi_{f,g} : A_f \to A_g$, which is the localization of A_f by inverting g.
(iii) Consequently if $U_f = U_g$ if and only if there exists an A-algebra isomorphism $A_f \to A_g$. When this happens, the A-algebra isomorphism $A_f \to A_g$ is unique, and we will use the induced identification, to simply write $A_f = A_g$.
(iv) Under the above identifications, show that the association $U_f \mapsto A_f$ defines a contrafunctor $U_f \mapsto A_f$ from the category of basic open subsets of $|\operatorname{Spec} A|$ with arrows as inclusions to the category of A-algebras and A-homomorphisms.

1.4 Exercise: Exactness. (1) Let $a_1, a_2 \in \mathbb{Z}$, $b_1, b_2 \in \mathbb{Z} - \{0\}$, such that b_1, b_2 are coprime, and $a_1/b_1 = a_2/b_2 \in \mathbb{Q}$. Then show that b_i divides a_i for $i = 1, 2$.
(2) Let A be a commutative ring (which is not necessarily a domain), and let $a_1, \ldots, a_n$, $b_1, \ldots, b_n \in A$ such that the ideal $(b_1, \ldots, b_n) = A$. Suppose that for each pair j, k of indices, we have

$$\frac{a_j}{b_j} = \frac{a_k}{b_k} \in A_{b_j b_k}.$$

Then show that there exists a unique element $x \in A$ such that for all $i = 1, \ldots, n$ we have

$$\frac{x}{1} = \frac{a_i}{b_i} \in A_{b_i}.$$

(3) Let $(f_i)_{i \in I}$ be an indexed collection of elements of a ring A, parameterized by a set I, such that the set $\{f_i \,|\, i \in I\} \subset A$ generates the unit ideal. Show that the following diagram of rings is exact:

$$A \to \prod_i A_{f_i} \rightrightarrows \prod_{(j,k)} A_{f_j f_k}.$$

In the above the first map sends $x \mapsto (x_i)_{i \in I}$ where $x_i = x/1 \in A_{f_i}$. The two maps $\phi_1, \phi_2 : \prod_i A_{f_i} \rightrightarrows \prod_{(j,k)} A_{f_j f_k}$ are defined as follows. The (j,k)th component of $\phi_1(a_i/f_i^{n_i})_{i \in I}$ is $a_j/f_j^{n_j}$, and the (j,k)th component of $\phi_2(a_i/f_i^{n_i})_{i \in I}$ is $a_k/f_k^{n_k}$.

1.5 All schemes and morphisms. A scheme is a locally ringed space $X = (|X|, \mathcal{O}_X)$ such that each $x \in X$ has an open neighbourhood U and there exists a ring A (depending on U) such that $(U, \mathcal{O}_X|_U)$ is isomorphic as a locally ringed space with the affine scheme $\operatorname{Spec} A = (|\operatorname{Spec} A|, \mathcal{O}_A)$. A morphism of schemes $f : X \to Y$ is a morphism of locally ringed spaces. Recall that this means $f = (|f|, f^\sharp)$ where $|f| : |X| \to |Y|$ is a continuous map of topological spaces and $f^\sharp : |f|^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ (equivalently by adjunction, $f^\sharp : \mathcal{O}_Y \to |f|_*\mathcal{O}_X$) is a homomorphism of sheaves of rings such that for each $x \in X$ the induced stalk level map $f^\sharp_x : \mathcal{O}_{Y,|f|(x)} \to \mathcal{O}_{X,x}$ is a local homomorphism of local rings.

Minimal primes, associated primes, zero divisors

1.6 The subset $S(A) \subset A$ of non zero-divisors. An element $a \in A$ is called a **non zero-divisor** if the scalar multiplication map $A \to A : x \mapsto ax$ is injective, in other words, the annihilator ideal $ann(a) = 0$. The set of all non zero-divisors in A will be denoted by $S(A) \subset A$. It is nonempty, and closed under multiplication, with $1 \in A^\times \subset S(A)$ where $A^\times$ is the multiplicative group of all invertible elements of A. If $A \to A : x \mapsto ax$ is not injective, that is, $ann(a) \neq 0$, then a is called a **zero divisor**, and all these form the subset denoted by $Zerodiv(A) = A - S(A)$.

1.7 Exercise. Show that the following are equivalent: (i) $A = 0$ (ii) $0 \in S(A)$ (iii) $Zerodiv(A) = \emptyset$ (iv) $S(A) = A$.

1.8 Exercise: Minimal primes. (1) Show that the following are equivalent for any prime ideal $\mathfrak{p}$ in a ring A.
(i) $\mathfrak{p}$ is minimal in the set of all prime ideals in A w.r.t. inclusion.
(ii) $\mathfrak{p}A_\mathfrak{p}$ is the nil radical of $A_\mathfrak{p}$.
(iii) $A_\mathfrak{p}$ is an artin local ring.
(2) Show that the intersection of any nested family of prime ideals in A is a prime ideal. Deduce by Zorn's lemma that any prime $\mathfrak{p}_0$ in a ring contains a prime $\mathfrak{p}_1 \subset \mathfrak{p}_0$ which is a minimal prime in A. In particular, any non-zero ring has at least one minimal prime ideal.

1.9 Exercise: Zero divisors in a reduced ring. (1) If $\mathfrak{p}$ is a minimal prime in A, then all elements of $\mathfrak{p}$ are zero divisors in A.
(2) Let A be **reduced**, that is, $Nil(A) = 0$. Then

$$Zerodiv(A) = \cup\{\mathfrak{p} \,|\, \mathfrak{p} \text{ is a minimal prime in } A\}.$$

1.10 Exercise: A minimal prime decomposition in a Noetherian ring. (1) If $I \subset A$ is an ideal in a commutative ring and $a, b \in A$ such that $ab \in I$, then $\sqrt{I} = \sqrt{I + (a)} \cap \sqrt{I + (b)}$.
(2) Use the above to show that any radical ideal $I = \sqrt{I}$ in a **noetherian** ring A can be expressed as a finite intersection $I = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_n$ of distinct prime ideals $\mathfrak{p}_i$ where $n \geq 0$ ($n = 0$ if and only if $I = A$), where the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ is **minimal** in the sense that I is not the intersection of any proper subset of it.
(3) Show that if A is noetherian and I any radical ideal in A, then a minimal prime decomposition $I = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_n$ as above is unique, and the primes $\mathfrak{p}_i$ are exactly all the the minimal primes in A that contain I (in other words, $\mathfrak{p}_i/I$ are exactly all the minimal primes in A/I).

1.11 Definition of an associated prime. A prime $\mathfrak{p}$ is said to be an **associated prime** for an A-module M if there exists an injective A-module homomorphism

$A/\mathfrak{p} \hookrightarrow M$, equivalently, if there exists $x \in M$ such that $ann(x) = \mathfrak{p}$. The set of all associated primes is denoted by $Ass_A(M)$. **Basic example.** If $\mathfrak{p} \subset A$ is any prime, then $Ass_A(A/\mathfrak{p}) = \{\mathfrak{p}\}$.

1.12 Every largest annihilator is an associated prime. Let $\mathcal{ANN}(M) = \{I \subset A \,|\, I = ann(x) \text{ for some } 0 \neq x \in M\}$. Note that if $M \neq 0$ then $\mathcal{ANN}(M) \neq \emptyset$, and $Ass_A(M) \subset \mathcal{ANN}(M)$. Any maximal element of $\mathcal{ANN}$ w.r.t. the inclusion partial order $I \subset J$ on $\mathcal{ANN}(M)$ is an associated prime of M (but not conversely: even though every associated prime of M is a member of $\mathcal{ANN}(M)$, it is not necessarily a maximal element of $\mathcal{ANN}(M)$). If $M \neq 0$ and A is noetherian, then $\mathcal{ANN}(M)$ is nonempty and therefore by noetherianness it has a maximal element, and hence $Ass_A(M) \neq \emptyset$.

1.13 Associated primes and localization. (1) If $S \subset A$ is a multiplicative subset of a ring, and $i : \operatorname{Spec} S^{-1}A \to \operatorname{Spec} A$ the map defined by contraction of primes. Then for any $S^{-1}A$-module N we have $Ass_A(N) = i(Ass_{S^{-1}A}(N))$.

(2) If A is noetherian, then for any A-module M, the following holds:

$$Ass_{S^{-1}A}(S^{-1}M) = \{S^{-1}\mathfrak{q} \,|\, \mathfrak{q} \in Ass(M) \text{ and } \mathfrak{q} \cap S = \emptyset\} = \{S^{-1}\mathfrak{q} \,|\, \mathfrak{q} \in Ass_A(S^{-1}M)\}.$$

1.14 $Zerodiv_A(M)$ **in terms of** $Ass_A(M)$**.** If A is noetherian then

$$Zerodiv_A(M) = \cup\, Ass_A(M).$$

1.15 Filtration and associated primes. Let A be a noetherian ring and let M be a finite A module. Then M has a finite filtration $M = M_0 \supset \ldots \supset M_n = 0$ by submodules such that each graded piece M_i/M_{i+1} is isomorphic $A/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$ in A (the $\mathfrak{p}_i$'s need not be distinct). For any such filtration, we have $ann(M) \subset \mathfrak{p}_i$ for each i, and moreover for every associated prime $\mathfrak{p}$ of M, there exists some $0 \leq i \leq n-1$ such that $\mathfrak{p}_i = \mathfrak{p}$ (that is, M_i/M_{i+1} is isomorphic $A/\mathfrak{p}$). But not conversely: each of the primes $\mathfrak{p}_i$ need not be associated to M. Example: Let $A = M = \mathbb{Z}$. This has $Ass(M) = \{(0)\}$. The filtration $\mathbb{Z} \supset 2\mathbb{Z} \supset 6\mathbb{Z} \supset 12\mathbb{Z} \supset 0$ has graded pieces $\mathbb{Z}/(2)$, $\mathbb{Z}/(3)$, $\mathbb{Z}/(2), \mathbb{Z}/(0)$.

1.16 Exercise. Show that a ring A is artinian if and only if it is noetherian of Krull dimension 0.

1.17 Krull's principal ideal theorem. Let A be a noetherian ring and let $a \in S(A) - A^\times$ (in words, a is neither a zero-divisor nor a unit in A). Let $\mathfrak{p}$ be a prime ideal of A that is minimal w.r.t. inclusion in the set of all prime ideals of A that contain a. Then $\mathfrak{p}$ has height 1, that is, $A_\mathfrak{p}$ is of dimension 1.

Meromorphic functions over locally noetherian schemes

1.18 Definition of the total quotient ring $Q(A)$ of a ring A. The localization $Q(A) = S(A)^{-1}A$, obtained by inverting the multiplicative subset $S(A)$ of all non zero-divisors in A, is called the total quotient ring of A. **Exercise:** The localization homomorphism $A \to Q(A)$ is injective and flat.

1.19 Exercise: Functoriality of S and Q for flat homomorphisms. (1) Show that if $\phi : A \to B$ is a flat homomorphism of rings, then $\phi(S(A)) \subset S(B)$.
(2) Deduce that if $\phi : A \to B$ is a flat homomorphism of rings, then there is a unique ring homomorphism $Q(\phi) : Q(A) \to Q(B)$ such that the following diagram commutes.

$$\begin{array}{ccc} A & \hookrightarrow & Q(A) \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle Q(\phi)} \\ B & \hookrightarrow & Q(B) \end{array}$$

(3) Show that if $\phi : A \to B$ is a flat homomorphism of rings, then $Q(\phi) : Q(A) \to Q(B)$ is flat.
(4) Let $\mathcal{F}$ be the category of all rings and flat homomorphisms. Show that S is a subfunctor of the forgetful functor $\mathcal{F} \to \mathcal{M} : (A, +, \times) \mapsto (A, \times)$ where $\mathcal{M}$ denotes the category of monoids, while Q is a functor from $\mathcal{F}$ to $\mathcal{F}$.

1.20 Exercise: The sheaf $\mathcal{A}^\times$. Let $(X, \mathcal{A})$ be a ringed space. For any open $U \subset X$ let $\mathcal{A}^\times(U) = \mathcal{A}(U)^\times$, the multiplicative group of units in the ring $\mathcal{A}(U)$. Show that
(1) If $V \subset U$ are open subsets of X and $f \in \mathcal{A}^\times(U) \subset \mathcal{A}(U)$ then $f|_V \in \mathcal{A}^\times(V)$.
(2) The resulting presheaf $\mathcal{A}^\times$ is a sheaf of groups under multiplication.
(3) If $x \in X$ then the homomorphisms $\mathcal{A}(U) \to \mathcal{A}_x$, where U varies over open neighbourhoods of x, induce an isomorphism $(\mathcal{A}^\times)_x \stackrel{\sim}{\to} (\mathcal{A}_x)^\times$.

1.21 The subsheaf $\mathcal{S} \subset \mathcal{A}$ on a locally ringed space $(X, \mathcal{A})$. Let $(X, \mathcal{A})$ be a ringed space, let $U \subset X$ be open, and let $f \in \mathcal{A}(U)$.
Exercise Show that the following are equivalent:
(i) For each open $V \subset U$, the restriction $f|_V$ is a non zero-divisor in the ring $\mathcal{A}(V)$ (that is, $f|_V \in S(\mathcal{A}(V))$).
(ii) For each $x \in U \subset X$, the germ f_x is a non zero-divisor in the stalk $\mathcal{A}_x$, that is, $f_x \in S(\mathcal{A}_x)$.
Definition. Let $\mathcal{S}(U) \subset \mathcal{A}(U)$ consist of all f that satisfy the above equivalent conditions (i) and (ii). In particular, $\mathcal{S}(U) \subset S(\mathcal{A}(U))$ (in some examples this inclusion can be proper). This defines a subsheaf of sets of $\mathcal{A}$ that is closed under multiplication, with $\mathcal{A}^\times \subset \mathcal{S}$. For any $x \in X$, the induced map $\mathcal{S}_x \to S(A_x)$ is injective, however, it need not be a bijection.

1.22 Definition: The sheaf $\mathcal{K}$ of $\mathcal{A}$-algebras on a locally ringed space $(X, \mathcal{A})$. A presheaf of $\mathcal{A}$-algebras on X is defined by $U \mapsto \mathcal{A}(U)[\mathcal{S}(U)^{-1}]$, the localization of $\mathcal{A}(U)$

by inverting $\mathcal{S}(U)$. The sheaf $\mathcal{K}$ of $\mathcal{A}$-algebras is by definition the sheafification of this presheaf.

1.23 Caution. For schemes, it is difficult to say what is $\mathcal{S}(U)$ and $\mathcal{K}(U)$ even for an affine scheme $U = \operatorname{Spec} A$ (the common sense candidates $S(A)$ and $Q(A)$ are not always correct). Also, it is difficult to say what are the stalks of $\mathcal{S}$ and $\mathcal{K}$ at a point x of a scheme X in general. (again, the common sense candidates $S(A_{X,x})$ and $Q(A_{X,x})$ are not always correct). As Kleiman pointed out in [K], both [EGA] and [H] have fallen prey to these errors. However, if X is locally noetherian, then these guesses are indeed correct, as shown in Mumford [M] Chapter 9 Section 1. The following lemma is needed for the proof in Mumford.

Lemma 1.24 *(1) For any ring A, let the subset $\mathcal{S}(A) \subset A$ consist of all $a \in A$ such that for every prime ideal $\mathfrak{p} \subset A$, the image of a under the localization homomorphism $A \to A_{\mathfrak{p}}$ is a non zero-divisor in $A_{\mathfrak{p}}$. Then $\mathcal{S}(A) \subset S(A)$. If the ring A is noetherian, then $\mathcal{S}(A) = S(A)$.*

(2) If A is noetherian and $I \subset A$ an ideal such that $Ann(I) = 0$ then $S(A) \cap I \neq \emptyset$, where $Ann(I) \subset A$ denotes the annihilator ideal of I.

(3) Let $f_1, \ldots, f_n$ generate the unit ideal in A. If A is noetherian then the diagrams

$$S(A) \to \prod_i S(A_{f_i}) \rightrightarrows \prod_{(j,k)} S(A_{f_j f_k}) \text{ and } Q(A) \to \prod_i Q(A_{f_i}) \rightrightarrows \prod_{(j,k)} Q(A_{f_j f_k}),$$

that are obtained by respectively applying the functors S and Q to the exact diagram $A \to \prod_i A_{f_i} \rightrightarrows \prod_{(j,k)} A_{f_j f_k}$ of rings and flat homomorphisms (see Ex.1.4.(3)), are exact diagrams (means in each diagram, the first map is injective, and its image equal the subset where the two other maps coincide).

Proof. (1) The inclusion $\mathcal{S}(A) \subset S(A)$ is obvious. If A is noetherian, its set of associated prime ideals $Ass(A)$ is finite and the set $Zerodiv(A)$ of all zero divisors in A equals $\cup\, Ass(A)$. If $\mathfrak{q}$ is any prime ideal in A, then $Ass(A_{\mathfrak{q}}) = \{\mathfrak{p}A_{\mathfrak{q}} \,|\, \mathfrak{p} \in Ass(A) \text{ and } \mathfrak{p} \subset \mathfrak{q}\}$. Hence if $a \in S(A) = A - Zerodiv(A) = A - \cup\, Ass(A)$, then $a/1 \in S(A_{\mathfrak{q}})$, showing $S(A) \subset \mathcal{S}(A)$.

(2) Suppose to the contrary that $I \subset Zerodiv(A) = \cup Ass(A)$. As $Ass(A)$ is a finite set of prime ideals, by prime avoidance there exists $\mathfrak{p} \in Ass(A)$ such that $I \subset \mathfrak{p}$. Being an associated prime, $\mathfrak{p}$ equals $Ann(x)$ the annihilator ideal of some nonzero element $x \in A$. Hence $x \in Ann(I)$ showing that $Ann(I) \neq 0$.

(3) The following is expanded from the proof sketched in Mumford [M] page 61. We first show that the ring homomorphism $u : Q(A) \to \prod_i Q(A_{f_i})$ is injective. Suppose that $a/b \in Q(A)$, where $a \in A$ and $b \in S_A$, maps to $0 \in Q(A_{f_i})$ for each i, equivalently, $a \mapsto 0 \in Q(A_{f_i})$. Hence for each i there exists some $x_i \in S(A_{f_i})$ such that $ax_i = 0 \in A_{f_i}$. Choosing a common large enough integer N, we can write $x_i = c_i/f_i^N$ for each i, where $c_i \in A$. Hence $ac_i f_i^M = 0 \in A$ for all i for some common large enough integer M. As $x_i = c_i/f_i \in S(A_{f_i})$ and as f_i is invertible in A_{f_i}, it follows that $c_i/1 \in S(A_{f_i})$. Hence

$a/1 = 0 \in A_{f_i}$ for each i, and so $a = 0$ as the f_i generate the unit ideal in A. This shows that the left hand side map is injective.

Clearly, the image of the first map lies inside the equalizer of the other two maps. We now prove that its image is indeed the equalizer.

Note that if $f \in A$, then $A \to A_f$ is flat, and the resulting map $Q(A) \to Q(A_f)$ is not necessarily surjective as we might have $f \in Zerodiv(A)$. However, every element of $Q(A_f)$ can be written as a/b where $a, b \in A$ are such that $b/1 \in S(A_f)$ (even though b itself may not lie in $S(A)$).

Suppose that $(a_i/b_i) \in \prod_i Q(A_{f_i})$ lies in the equalizer, where $a_i, b_i \in A$ and $b_i/1 \in S(A_{f_i})$. Being in the equalizer means $a_i/b_i = a_j/b_j \in Q(A_{f_if_j})$ for all i, j, equivalently (as $S(A_{f_if_j})$ consists of non zero-divisors), $a_ib_j = a_jb_i \in A_{f_if_j}$ for all i, j. Hence, there exists a common large integer N such that $a_ib_jf_i^Nf_j^N = a_jb_if_i^Nf_j^N \in A$ for all i, j. Replacing each a_i by $a_if_i^N$ and b_i by $b_if_i^N$ leaves a_i/b_i unchanged in $Q(A_{f_i})$, but now $a_ib_j = a_jb_i \in A$ for all i, j, with $b_i/1 \in S(A_{f_i})$ for each i.

We want to show that there exist $\alpha \in A$ and $\beta \in S(A)$ such that $\alpha/\beta = a_i/b_i \in Q(A_{f_i})$ for each i. This means $\alpha b_i = \beta a_i \in A_{f_i}$. Let $I \subset A$ be defined by

$$I = \{\, z \in A \,|\, za_i \in b_iA_{f_i} \text{ for each } i \,\}.$$

Then I is an ideal in A. Moreover, each b_j lies in I, as $b_ja_i = a_jb_i \in (b_i) \subset A$.

We now show that $Ann(I) = 0$, where $Ann(I) = \{\, c \in A \,|\, cI = 0 \,\}$. If $c \in Ann(I)$ then $cb_i = 0$ for each i. But $b_i/1 \in S(A_{f_i})$, hence $c/1 = 0 \in A_{f_i}$ for each i. Hence $c = 0$.

Therefore by statement (2) of the lemma, $I \not\subset Zerodiv(A)$, that is, $I \cap S(A) \neq \emptyset$. Choose any element $\beta \in I \cap S(A)$. Then $\beta a_i = b_iy_i$ for some $y_i \in A_{f_i}$. Hence $\beta\, a_i/b_i = y_i \in A_{f_i} \subset Q(A_{f_i})$. As by assumption $a_i/b_i \in Q(A_{f_i})$ and $a_j/b_j \in Q(A_{f_j})$ have the same restriction to $Q(A_{f_if_j})$, the elements $\beta\, a_i/b_i \in A_{f_i}$ and $\beta\, a_j/b_j \in A_{f_j}$ have the same restriction to $A_{f_if_j}$. Hence by the sheaf property of $\mathcal{O}_X$, there exists a unique $\alpha \in A$ such that $\beta\, a_i/b_i = \alpha/1 \in A_{f_i}$ for each i.

The resulting element $\alpha/\beta \in Q(A)$ maps to $(a_i/b_i) \in \prod_i Q(A_{f_i})$. □

1.25 Exercise. Let X be a locally noetherian scheme, that is, it has an affine open covering (U_i) where each U_i is isomorphic to $\operatorname{Spec} A_i$ for a noetherian ring A_i. Show that any affine open subscheme of X is isomorphic to $\operatorname{Spec} A$ for a noetherian ring A.

1.26 The subsheaf $\mathcal{S}_X \subset \mathcal{O}_X$ on a locally noetherian scheme X. We follow the strategy of defining $\mathcal{S}_X$ on affine schemes $\operatorname{Spec} A$ in terms of defining $\mathcal{S}_X(U_f)$ for $U_f = \operatorname{Spec} A_f$ for any $f \in A$, and defining restrictions $\mathcal{S}_X(U_f) \to \mathcal{S}_X(U_{fg})$ for any $f, g \in A$, and then appealing to Exercise 1.1 and to the Lemma 1.24.(1) (which needs the noetherian condition) to complete the construction of the sheaf $\mathcal{S}_X$ on any locally noetherian scheme X. We define $\mathcal{S}_X(U_f) = S(A_f)$, and we define the restriction map $\mathcal{S}_X(U_f) \to \mathcal{S}_X(U_{fg})$ to be the map $S(A_f) \to S(A_{fg})$ induced by the localization homomorphism $A_f \to A_{fg} = (A_f)_g$ which is flat.

1.27 The sheaf $\mathcal{K}_X$ of $\mathcal{O}_X$-algebras on a locally noetherian scheme X. For any affine open $U = \operatorname{Spec} A \subset X$ (note that A is noetherian by Exercise 1.25) we define $\mathcal{K}_X(U) = Q(A)$ as an A-algebra. For $f, g \in A$ we define the restriction map $\mathcal{K}_X(U_f) \to \mathcal{K}_X(U_{fg})$ to be the map $Q(A_f) \to Q(A_{fg})$ induced by the localization homomorphism $A_f \to A_{fg} = (A_f)_g$ which is flat. By Exercise 1.1 and Lemma 1.24.(3) (which needs the noetherian condition) we get a sheaf $\mathcal{K}_X$ of $\mathcal{O}_X$-algebras.

1.28 Exercise. If $\operatorname{Spec} B \subset \operatorname{Spec} A$ are affine open subschemes of a locally noetherian scheme X, then the corresponding ring homomorphism $A \to B$ is flat, so it maps $S(A)$ into $S(B)$ and induces a homomorphism $Q(A) \to Q(B)$. Show that these are the restriction maps from $\operatorname{Spec} A$ to $\operatorname{Spec} B$ for the sections of the sheaves $\mathcal{S}_X$ and $\mathcal{K}_X$.

1.29 Exercise. If X is a locally noetherian scheme and $x \in X$, show that we have natural isomorphisms $\mathcal{S}_{X,x} \to S(A_{X,x})$ and $\mathcal{K}_{X,x} \to Q(A_{X,x})$ at the level of stalks.

1.30 Exercise. For a locally noetherian scheme X, let $\mathcal{K}_X^\times$ denote the sheaf of multiplicative groups formed by the units in $\mathcal{K}_X$ (see Exercise 1.20). Show that it satisfies $\mathcal{K}_X^\times(\operatorname{Spec} A) = Q(A)^\times$ and $(\mathcal{K}_X^\times)_x = \mathcal{K}_{X,x}^\times = Q(\mathcal{O}_{X,x})^\times = Q(A_\mathfrak{p})^\times$ for any affine open subscheme $\operatorname{Spec} A \subset X$ and point $x = \mathfrak{p} \in \operatorname{Spec} A$.

1.31 Exercise. Let X be an integral scheme. By definition, the **function field** $K(X)$ of X is the stalk $\mathcal{O}_{X,\xi}$ of $\mathcal{O}_X$ at the generic point ξ of X. Show that taking the germ at ξ of a global section of $\mathcal{K}_X$ defines an isomorphism $\Gamma(X, \mathcal{K}_X) \to K(X)$. Moreover, it induces an isomorphism $\Gamma(X, \mathcal{K}_X^\times) \to K(X)^\times$.
Notation. When no ambiguity results, we will denote $\Gamma(X, \mathcal{K}_X)$ and $\Gamma(X, \mathcal{K}_X^\times)$ simply by $\mathcal{K}(X)$ and $\mathcal{K}(X)^\times$. Similarly we will denote $\Gamma(X, \mathcal{S}_X)$ simply by $\mathcal{S}(X)$, and the stalks $\mathcal{S}_{X,x}, \mathcal{K}_{X,x}, \mathcal{K}_{X,x}^\times$simply by $\mathcal{S}_x, \mathcal{K}_x, \mathcal{K}_x^\times$.

1.32 Flat pullbacks. If $f : X \to Y$ is a flat morphism of locally noetherian schemes, then as flat homomorphisms of rings preserve non zero-divisors, the homomorphism $f^\sharp : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ induces homomorphisms of sheaves (which we again denote simply by) $f^\sharp : f^{-1}\mathcal{S}_Y \to \mathcal{S}_X$, $f^\sharp : f^{-1}\mathcal{K}_Y \to \mathcal{K}_X$ and $f^\sharp : f^{-1}\mathcal{K}_Y^\times \to \mathcal{K}_X^\times$. If $f : X \to Y$ and $g : Y \to Z$ are both flat then $f^\sharp \circ (f^{-1}(g^\sharp)) = (g \circ f)^\sharp$ on $\mathcal{S}$, $\mathcal{K}$ and $\mathcal{K}^\times$.

1.33 Support $\operatorname{supp}(s)$ **of a section** s **of a sheaf of abelian groups.** Let X be a topological space, $\mathcal{F}$ a sheaf of sets on X, and let $s, t \in \Gamma(X, \mathcal{F})$. Let $s_x, t_x \in \mathcal{F}_x$ denote the germs of s and t in the stalk $\mathcal{F}_x$ for any $x \in X$. Then the set $\{x \in X \mid s_x = t_x\}$ is open in X (exercise). In particular, let $\mathcal{F}$ be a sheaf of abelian groups, and we take $t = 0$ in the above. This shows that the set

$$\operatorname{supp}(s) = \{x \in X \mid s_x \neq 0\}$$

is closed in X. It is called as the **support** of s.

1.34 The sheaf of Cartier divisors. On any scheme X, we have a short-exact sequence of sheaves of abelian groups $0 \to \mathcal{O}_X^\times \to \mathcal{K}_X^\times \to \mathcal{K}_X^\times/\mathcal{O}_X^\times \to 0$. The quotient sheaf $\mathcal{K}_X^\times/\mathcal{O}_X^\times$ is called the sheaf of Cartier divisors on X. A **Cartier divisor** on X is a global section $D \in \Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$. The **support** $|D|$ of the Cartier divisor D is the closed subset of $|X|$ which is the set-topological support

$$|D| = \mathrm{supp}(D)$$

(see stmt 1.33) of the global section D of the sheaf $\mathcal{K}_X^\times/\mathcal{O}_X^\times$.

1.35 The Cartier divisor $\mathrm{div}(r)$ **of an invertible meromorphic function** r**.** The image of any $r \in \Gamma(X, \mathcal{K}_X^\times)$ under the map $\Gamma(X, \mathcal{K}_X^\times) \to \Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$ induced by the quotient $\mathcal{K}_X^\times/ \to \mathcal{K}_X^\times/\mathcal{O}_X^\times$ is denoted by $\mathrm{div}(r)$, and is it is called the Cartier divisor of r.

1.36 The support $|r|$ **of an invertible meromorphic function** r**, and its codimension.** By the support $|r|$ of $r \in \Gamma(X, \mathcal{K}_X^\times)$, we will mean the support

$$|r| = \mathrm{supp}(\mathrm{div}(r))$$

which is a closed subset of $|X|$.
Caution: Thus, $|r|$ contains – but may not be equal to – the support $\mathrm{supp}(r)$ of the section r of the sheaf $\mathcal{K}_X^\times$.
If $U = X - |r|$ is regarded as an open subscheme of X, then $r|_U \in \Gamma(U, \mathcal{O}_U^\times)$. If $X = \mathrm{Spec}\, A$ where A is noetherian, then we can write $r = s/t$ where $s, t \in S(A)$, and hence $|r| \subset V(a) \cup V(b)$ where $V(a), V(b)$ are the closed subsets defined by $a = 0$ and $b = 0$. If $\mathfrak{p}$ is a minimal prime of A, then $\mathfrak{p} \subset Zerodiv(A)$, and so $a \notin \mathfrak{p}$, $b \notin \mathfrak{p}$, showing that $\mathfrak{p} \in U = X - |r|$. For any $r \in \Gamma(X, \mathcal{K}_X^\times)$ for a locally noetherian scheme X, the above argument shows that $|r|$ does not contain any height 0 points of X.

1.37 The restriction of $r \in \mathcal{K}^\times(X)$ **to an irreducible component** Y **of** X**.** Let X be a noetherian scheme. If $Y \subset X$ is an irreducible component of X, then the generic point $\eta \in Y$ has height 0 in X, that is, $\dim(\mathcal{O}_{X,\eta}) = 0$. Hence $\eta \notin |r|$, and so $U \cap Y \neq \emptyset$ where $U \subset X$ is the open subscheme $X - |r|$, and therefore the regular function $r|_U \in \Gamma(U, \mathcal{O}_U^\times)$ further restricts to a section $r_Y \in \Gamma(Y \cap U, \mathcal{O}_Y^\times)$. As Y is integral, $\Gamma(Y \cap U, \mathcal{O}_Y^\times) \subset K(Y)^\times$ where $K(Y)$ denotes the function field of Y, which in turn equals $\mathcal{K}_Y(Y)$. Hence we have a restriction homomorphism $res_{X,Y} : \mathcal{K}_X^\times(X) \to \mathcal{K}_Y^\times(Y) : r \mapsto r_Y$ for any irreducible component Y of X. If $U \subset X$ is an open subscheme such that $Y \cap U \neq \emptyset$, then $Y \cap U$ is an irreducible component of U, and it follows from the above definition of restrictions that $r_Y|_{Y\cap U} = (r|_U)_{Y\cap U} \in K_{Y\cap U}(Y \cap U)$, an equality that will be useful later. We will denote $r_Y|_{Y\cap U}$ and $(r|_U)_{Y\cap U}$ by the common symbol $r|_{Y\cap U}$.

2 Orders of zeros and poles

Finite length modules and the index of an endomorphism

2.1 Extended natural numbers $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. We adjoin an extra formal element ∞ (called 'infinity') to the set $\mathbb{N} = \{0, 1, 2, \ldots\}$ of all natural numbers to obtain the set $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. It is totally ordered by extending to $\overline{\mathbb{N}}$ the usual ordering $\leq$ on $\mathbb{N}$ by putting $x \leq \infty$ for all $x \in \overline{\mathbb{N}}$. It becomes a commutative monoid by extending to $\overline{\mathbb{N}}$ the usual addition $+$ on $\mathbb{N}$ by putting $x + \infty = \infty + x = \infty$ for all $x \in \overline{\mathbb{N}}$.

2.2 Finite length modules. A module M over a commutative ring A is said to be **simple** if $M \neq 0$ and there is no submodule N of M such that $0 \neq N \neq M$. A **composition series** for an A-module M is a finite filtration $M = M_0 \supset M_1 \supset \ldots \supset M_n = 0$ where $n \geq 0$, such that each of the n graded pieces $M_0/M_1, \ldots, M_{n-1}/M_n$ is a simple A-module. The number n is called the length of the composition series. By definition, the zero module has a composition series of length $n = 0$. The **Jordan-Holder theorem** says that if an A-module M has at least one composition series, then
(i) any two composition series for M have the same length,
(ii) any strictly decreasing filtration of M is finite and can be refined (by inserting some extra submodules if needed) to a composition series for M, and
(iii) the isomorphism classes of graded pieces and their multiplicities of occurrence are the same in any two composition series for M (only the order of occurrence of the graded pieces may possibly be permuted).

We say that M has **finite length** if M admits a composition series.

2.3 The length $\ell_A(M) \in \overline{\mathbb{N}}$ of an arbitrary module. If an A-module $M \neq 0$ has finite length, then by Jordan-Holder theorem any two composition series for M have the same length $n \in \mathbb{N}$, which we call as the **length** of the module M, and we denote it by $\ell_A(M) \in \mathbb{N} \subset \overline{\mathbb{N}}$. Note that $M = 0$ if and only if $\ell_A(M) = 0$, and M is simple if and only if $\ell_A(M) = 1$. We say that M has **infinite length** if M does not have any composition series, and we put $\ell_A(M) = \infty \in \overline{\mathbb{N}}$ in that case. Thus, the length $\ell_A(M)$ is always defined as an extended natural number. Note that $\ell_A(M) = \infty$ if and only if for each $n \in \mathbb{N}$, there exists a finite filtration of M by submodules $M = M_0 \supset M_1 \supset \ldots \supset M_n = 0$ where each of the n inclusions is proper.

We will sometimes drop the mention of A and simply write $\ell(M)$, if A is understood.

2.4 Exercise: additivity. If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of A-modules, then show that $\ell_A(M) = \ell_A(M') + \ell_A(M'')$ in terms of the binary operation $+$ on $\overline{\mathbb{N}}$ defined above. Hence M is of finite length if and only if both M' and M'' are so.

2.5 Exercise: Comparison with dimension of vector spaces. Recall that any two bases of a vector space V over a field k have the same cardinality, which is called the dimension of V over k (notation: $\dim_k(V)$). Show that V is finite dimensional if and only if V is of finite length over k, and in this case, $\dim_k(V) = \ell_k(V)$, and V is infinite dimensional (of some infinite cardinality) if and only if $\ell_k(V) = \infty$.

2.6 Exercises. (1) Up to isomorphism, what are the simple modules over a ring A?
(2) For any ring A, show that $\ell_A(A) < \infty$ if and only if A is artinian.
(3) If $(A, \mathfrak{m})$ is a local algebra over a field k such that the composite map $k \to A \to A/\mathfrak{m}$ is an isomorphism, then show that $\ell_A(M) = \ell_k(M)$ for any A-module M. Is this true if $k \to A/\mathfrak{m}$ is not an isomorphism?

2.7 Length and index. If $\phi : M \to M$ is an endomorphism of an A-module M such that both $\ker(\phi)$ and $\mathrm{coker}(\phi)$ are of finite length, then we say that ϕ is of finite index, and in that case we define the index $e_A(\phi, M) \in \mathbb{N}$ of ϕ by

$$e_A(\phi, M) = \ell_A(\mathrm{coker}(\phi)) - \ell_A(\ker(\phi)).$$

Exercise. If M is of finite index then $e_A(\phi, M) = 0$ for all $\phi : M \to M$.

2.8 Composites. *Let $\phi, \psi \in End_A(M)$ each have finite index. Then the composite $\phi\psi$ also has finite index and*

$$e_A(\phi\psi, M) = e_A(\phi, M) + e_A(\psi, M)$$

Proof. This follows from the exact sequence

$$0 \to \ker(\psi) \to \ker(\phi\psi) \stackrel{\psi}{\to} \ker(\phi) \to \mathrm{coker}(\psi) \stackrel{\phi}{\to} \mathrm{coker}(\phi\psi) \to \mathrm{coker}(\phi) \to 0,$$

as the alternating sum of lengths is zero for any exact sequence of modules of finite length which starts and ends with 0 modules. $\square$

2.9 Additivity of index. *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of A-modules, and $\phi \in End_A(M)$ such that $\phi(M') \subset M'$. Let $\phi' \in End_A(M')$ and $\phi'' \in End_A(M'')$ be induced by ϕ. If any two of ϕ, ϕ' and ϕ'' are of finite index then so is the remaining one. In that case we have*

$$e_A(\phi, M) = e_A(\phi', M') + e_A(\phi'', M'')$$

Proof. We have the following commutative diagram with short exact rows.

$$\begin{array}{ccccccc}
0 \to & M' & \to & M & \to & M'' & \to 0 \\
 & \downarrow \phi' & & \downarrow \phi & & \downarrow \phi'' & \\
0 \to & M' & \to & M & \to & M'' & \to 0
\end{array}$$

Applying Snake lemma gives us the exact sequence

$$0 \to \ker(\phi') \to \ker(\phi) \to \ker(\phi'') \to \mathrm{coker}(\phi') \to \mathrm{coker}(\phi) \to \mathrm{coker}(\phi'') \to 0.$$

The conclusion follows. $\square$

2.10 Corollary. *Let $\phi : M \to M$ be an endomorphism of an A-module. Let $N \subset M$ be an A-submodule such that $\phi(N) \subset N$ and $\ell_A(M/N) < \infty$. Then ϕ is of finite index if and only if $\phi|_N$ is of finite index, and in that case we have $e_A(\phi|_N, N) = e_A(\phi, M)$.*

Proof. As M/N is of finite length, $e_A(\phi'', M/N) = 0$ where $\phi'' : M/N \to M/N$ is induced by ϕ. The conclusion follows by statement **2.9**. $\square$

2.11 [Fulton lemma A.2.8] *Let A be a ring, M an A-module and ϕ, $\psi \in End(M)$ commuting endomorphisms of M such that both ϕ and ψ are injective. Let $\overline{\phi} \in End(\mathrm{coker}(\psi))$ and $\overline{\psi} \in End(\mathrm{coker}(\phi))$ be induced by ϕ and ψ. Then $\overline{\phi}$ is of finite index if and only if $\overline{\psi}$ is of finite index, and in that case*

$$e_A(\overline{\phi}, \mathrm{coker}(\psi)) = e_A(\overline{\psi}, \mathrm{coker}(\phi)).$$

Proof. Applying Snake lemma to the commutative diagram

$$\begin{array}{ccccccc}
0 \to & M & \stackrel{\phi}{\to} M \to & \mathrm{coker}(\phi) & \to 0 \\
 & \downarrow \psi & \downarrow \psi & \downarrow \overline{\psi} & \\
0 \to & M & \stackrel{\phi}{\to} M \to & \mathrm{coker}(\phi) & \to 0
\end{array}$$

gives the short exact sequence $0 \to \ker(\overline{\psi}) \to \mathrm{coker}(\psi) \stackrel{\overline{\phi}}{\to} \mathrm{coker}(\psi) \to \mathrm{coker}(\overline{\psi}) \to 0$ from which we get $\ker(\overline{\phi}) \cong \ker(\overline{\psi})$ and $\mathrm{coker}(\overline{\phi}) \cong \mathrm{coker}(\overline{\psi})$. $\square$

2.12 Example (Srinivas) Let $A = k + \mathfrak{m}^2 \subset k[x, y]$ where $\mathfrak{m} = (x, y) \subset k[x, y]$. Note that A is generated as a k-algebra by the finitely many generators x^2, xy, y^2, x^3, x^2y, xy^2, y^3. In the statement **2.11**, take $M = A$ and let $\phi = x^2$ and $\psi = y^2$ act by scalar multiplication A. The point of this example is that $\overline{\psi} = \overline{y}^2$ is not injective on $\mathrm{coker}(\phi) = A/(x^2)$, as it annihilates each element of $\mathfrak{n} = \mathfrak{m} \cap A \subset k[x, y]$. In fact, $ann(\overline{y}^2) = \mathfrak{n}$, so $\mathfrak{n}$ is an associated prime to $A/(x^2)$. Hence in the stmt 2.11, the index of $\overline{\phi}$ (or that of $\overline{\psi}$) cannot be replaced by the length of its cokernel alone (or by the order of vanishing of an element in this example).

2.13 Direct image and length. *Let $A \to B$ be a local homomorphism of noetherian local rings, and let M be a B-module. If M is of finite length over A then M is of finite length over B. For the reverse implication to hold for all M (or even for a single nonzero M which is of finite length over B), it is necessary and sufficient to assume that the residue field extension $(B/\mathfrak{n})/(A/\mathfrak{m})$ is finite, that is, $[B/\mathfrak{n} : A/\mathfrak{m}] < \infty$, where $\mathfrak{m} \subset A$ and $\mathfrak{n} \subset B$ denote the respective maximal ideals. Under this condition, any B-module M of finite length over B is of finite length over A, and moreover,*

$$\ell_A(M) = \ell_B(M)[B/\mathfrak{n} : A/\mathfrak{m}].$$

Proof. If a B-module M is of finite length over A, then it satisfies both the ascending and the descending chain conditions for A-submodules of M. Every B-submodule of M is also an A-submodule of M, and hence M satisfies acc and dcc for B-submodules. Hence M is of finite length over B. Now we consider the reverse implication. The B-module $B/\mathfrak{n}$ has length $\ell_B(B/\mathfrak{n}) = 1$, while $\ell_A(B/\mathfrak{n}) = \ell_{A/\mathfrak{m}}(B/\mathfrak{n}) = [B/\mathfrak{n} : A/\mathfrak{m}]$. Moreover, $B/\mathfrak{n}$ is up to isomorphism the only length 1 module (equivalently, simple module) over B, so if $\ell_B(M) < \infty$ then in any composition series for M the graded pieces are isomorphic to $B/\mathfrak{n}$. As $M \neq 0$, there is at least one such graded piece. Hence the assumption that $[B/\mathfrak{n} : A/\mathfrak{m}] < \infty$ is necessary for the reverse implication. We now show that it is sufficient. Let M be a B module with $\ell_B(M) < \infty$. Both sides of the equation $\ell_A(M) = \ell_B(M)[B/\mathfrak{n} : A/\mathfrak{m}]$ are additive over short exact sequences of B-modules, so by taking a composition series for M over B (in which the graded pieces are each of length 1 hence isomorphic to $B/\mathfrak{n}$), it is enough to prove the statement when $M = B/\mathfrak{n}$. But this is just the statement $\ell_A(B/\mathfrak{n}) = \ell_{A/\mathfrak{m}}(B/\mathfrak{n}) = [B/\mathfrak{n} : A/\mathfrak{m}]$. □

2.14 Flat pullback and length. *Let $A \to B$ be a local homomorphism of noetherian local rings, such that B is flat over A. Then the following holds:*

(1) B is faithfully flat over A, the map $A \to B$ is injective and the map $\operatorname{Spec} B \to \operatorname{Spec} A$ *is surjective.*

(2) Moreover, if A and B are artinian and if M is an A-module of finite length, then $B \otimes_A M$ is a B-module of finite length, and

$$\ell_B(B \otimes_A M) = \ell_A(M)\ell_B(B/\mathfrak{m}_A B)$$

where $\mathfrak{m}_A \subset A$ is the maximal ideal.

Proof. (1) Injectivity of $A \to B$: Suppose $0 \neq a \in \ker(A \to B)$. Then $ann_A(a) \neq A$, so $ann_A(a) \subset \mathfrak{m}_A$ as A is local. We have an injection of A-modules $\phi : A/ann_A(a) \to A : 1 \mapsto a$. As B is flat over A, the map $\phi \otimes 1_B : B/ann_A(a)B \to B$ is injective. But as $a \mapsto 0$, we see that $\phi \otimes 1_B = 0$. Hence $ann_A(a)B = B$. But as $A \to B$ is local and $ann_A(a) \subset \mathfrak{m}_A$, this is a contradiction. Hence $A \to B$ is injective.

Surjectivity of $\operatorname{Spec} B \to \operatorname{Spec} A$: Let $\mathfrak{p} \subset A$ be a prime. Then $A/\mathfrak{p} \to B/\mathfrak{p}B$ is a flat local homomorphism, so it is injective. So, we are reduced to proving that if $A \to B$ is a flat local injective homomorphism where A is a domain, then $K \otimes_A B \neq 0$ where K is the quotient field of A. Now, if $K \otimes_A B = 0$, then as $K = (A - \{0\})^{-1}A$, we see that $(A - \{0\})^{-1}B = 0$. As $1 = 0$ there exists $a \in A - \{0\}$ such that $a \cdot 1 = 0$ in B. But this contradicts the injectivity of $A \to B$. As $K \otimes_A B \neq 0$, the generic fiber of $\operatorname{Spec} B \to \operatorname{Spec} A$ is non-empty as desired.

Faithful flatness of $A \to B$: In general, for any ring homomorphism $f : A \to B$, faithful flatness is equivalent to flatness plus surjectivity of $\operatorname{Spec} B \to \operatorname{Spec} A$. If M is a non-zero A-module, then there exists a maximal ideal $\mathfrak{m} \subset A$ such that $A/\mathfrak{m}$ is the quotient of a submodule (of the form $A/\operatorname{ann}_A(x)$) of M. If B is flat over A, this implies that $B \otimes_A (A/\mathfrak{m})$ is a sub-quotient of $B \otimes_A M$. If $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective then each $B \otimes_A (A/\mathfrak{m})$ is non-zero, so $B \otimes_A M$ is non-zero. Hence B is faithfully flat over A. Conversely, if B

is faithfully flat over A then given any $\mathfrak{p} \in \operatorname{Spec} A$, the map $A_{\mathfrak{p}} \to B_{\mathfrak{p}} = (A - \mathfrak{p})^{-1}B$ is faithfully flat. Hence tensoring with $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ we see that $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}} \neq 0$, so $\mathfrak{p}$ is in the image of $\operatorname{Spec} B \to \operatorname{Spec} A$ showing surjectivity.

(2) Let M be an A-module of finite length n. Let $M = M_0 \supset \ldots \supset M_n = 0$ be a composition series. As $A/\mathfrak{m}_A$ is the only A-module of length 1, each $M_i/M_{i+1} \simeq A/\mathfrak{m}_A$ for $0 \leq i < n$. Applying $B \otimes_A -$ (which is exact as B is flat over A), we get a filtration $B \otimes_A M = B \otimes_A M_0 \supset \ldots \supset B \otimes_A M_n = 0$ with each $B \otimes_A (M_i/M_{i+1}) \simeq B/\mathfrak{m}_A B$. Applying $\ell_B(-)$ to the above filtration of $B \otimes_A M$, the equality $\ell_B(B \otimes_A M) = \ell_A(M)\ell_B(B/\mathfrak{m}_A B)$ follows. $\square$

Order of a meromorphic function

2.15 Motivation for the definition of order of vanishing. Let f be a holomorphic function of the variable z in a neighbourhood of $0 \in \mathbb{C}$. Then f can be locally represented around $z = 0$ by a convergent power series $\sum_{i \geq 0} a_i z^i$. Let $\mathbb{C}\{z\}$ denote the $\mathbb{C}$-algebra of all convergent power series in z. This is a discrete valuation ring with maximal ideal (z) and residue field $\mathbb{C}$. The residue homomorphism sends $f \mapsto f(0)$. The function f, if not identically zero, can be uniquely written as $z^n g$ where $n \in \mathbb{N}$, $g \in \mathbb{C}\{z\}$ and $g(0) \neq 0$. We say that f has a zero of order n at 0. The ring $\mathbb{C}\{z\}$ is a discrete valuation ring, equivalently, it is a noetherian local domain of Krull dimension 1 which is normal. The valuation ν on $\mathbb{C}\{z\} - \{0\}$ is defined by $\nu(f) = n$ if $f = z^n g$ where $g \in \mathbb{C}\{z\}$ and $g(0) \neq 0$. Equivalently, $\nu(f)$ is the unique natural number such that $f \in (z)^{\nu(f)} - (z)^{\nu(f)+1}$. In what follows, we want to replace the dvr $\mathbb{C}\{z\}$ by any noetherian local ring A of Krull dimension 1, and define an 'order of vanishing' $\operatorname{ord}_A(a)$ for some suitable elements $a \in A$. It turns out that we can do this very usefully for all non zero-divisors $a \in S(A)$. The rings A that we have in mind are the local rings at points of codimension 1 on an algebraic scheme (the generic points of codimension 1 closed subvarieties V of algebraic schemes). These rings are clearly noetherian local of Krull dimension 1. However, the scheme X may not be reduced around the generic point of V, or X may have more than one irreducible component that contains the subvariety V, and so the local ring A may not be a domain. Even if X is a variety, so that A is a noetherian local domain of Krull dimension 1, it may not be a discrete valuation ring as X might be singular along V. The required generalization of the order function is made using the the lengths of rings and modules.

2.16 Proposition : when is $\operatorname{ord}_A(a)$ finite? *Let A be a noetherian local ring of Krull dimension 1. Then the following two statements are equivalent for any $a \in A$.*
(1) The element a does not lie in any minimal prime $\mathfrak{p}$ of A.
(2) The local ring $A/(a)$ is of finite length, that is, $A/(a)$ is an artin local ring.
In particular, if $a \in S(A)$ then $A/(a)$ is an artin local ring.

Proof. As $\dim(A) = 1$, its maximal ideal $\mathfrak{m}$ is not a minimal prime, and the only prime ideals of A are $\mathfrak{m}$ and the minimal primes of A. Hence if (1) holds, then $\mathfrak{m}$ is the only prime ideal of A that contains a, so we must have $\sqrt{(a)} = \mathfrak{m}$. Hence there exists some $n \geq 1$ such that $\mathfrak{m}^n \subset (a)$. This shows $A/(a)$ is a quotient of $A/\mathfrak{m}^n$, which is artinian, so

$A/(a)$ is artinian, showing (2). If on the other hand if $a \in \mathfrak{p}$ for a minimal prime $\mathfrak{p}$, then the 1-dimensional noetherian local ring $A/\mathfrak{p}$ is a quotient of $A/(a)$, so $A/(a)$ cannot be artinian, and hence (2) implies (1).

Finally, as $S(A) \subset A - \mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_r$ where $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ (where $r \geq 1$) are the minimal primes of A, it follows that if $a \in S(A)$ then $A/(a)$ is an artin local ring. $\square$

2.17 Order of vanishing generalized from a dvr to a 1-dim noetherian local ring. Let A be a dvr with maximal ideal $\mathfrak{m}$. Then for any element $a \neq 0$ of A there exists a unique natural number $\nu(a) \geq 0$ such that $a \in \mathfrak{m}^{\nu(a)} - \mathfrak{m}^{\nu(a)+1}$. The corresponding map $\nu : A - \{0\} \to \mathbb{N}$ defined by $a \mapsto \nu(a)$ is called the **discrete valuation** on A. For any $a, b \in A - \{0\}$ (and so $ab \in A - \{0\}$ because A is a domain) we have $\nu(ab) = \nu(a) + \nu(b)$. The valuation ν also satisfies a (non-archimedean) triangle inequality, but that will not play any role for us, and anyway, it does not generalize to all 1-dim noetherian local rings (see example below). As $0 \in \mathfrak{m}^n$ for each $n \geq 0$, we can extend $\nu : A - \{0\} \to \mathbb{N}$ to a map $\nu : A \to \mathbb{N} \cup \{\infty\}$ by putting $\nu(0) = \infty$.

The generalization from a dvr to a 1-dim noetherian local ring is based on the observation that if A is a dvr and $a \in A - \{0\}$, then in fact $(a) = \mathfrak{m}^{\nu(a)}$, and so the ring $A/(a)$ is artinian with $\ell(A/(a)) = \nu(a)$. On the other hand, for $a = 0$, we have $\ell(A/(0)) = \infty = \nu(0)$.

The relationship between ν and length motivates the definition of an **order function** $\mathrm{ord}_A : A \to \mathbb{N} \cup \{\infty\}$ defined by $\mathrm{ord}_A(a) = \ell(A/(a))$ for any 1-dimensional noetherian local ring $(A, \mathfrak{m})$ (which may not even be a domain). By Proposition 2.16, $\mathrm{ord}_A(a)$ is finite if $a \in S(A)$ while $\mathrm{ord}_A(a) = \infty$ if $a \in \mathfrak{p}$ for some minimal prime $\mathfrak{p}$. Moreover, if $a, b \in S(A)$ then $ab \in S(A)$, and it can be seen that $\mathrm{ord}(ab) = \mathrm{ord}(a) + \mathrm{ord}(b)$ (see Exercise 2.18).

The map $\mathrm{ord}_A : S(A) \to \mathbb{N}$ takes multiplication to addition, so it extends to a well-defined group homomorphism

$$\mathrm{ord}_A : Q(A)^{\times} \to \mathbb{Z}$$

by putting $\mathrm{ord}_A(a/b) = \mathrm{ord}(a) - \mathrm{ord}(b)$ for any $a, b \in S(A)$.

Caution. For a dvr, we have $\nu(a + b) \geq min\{\nu(a), \nu(b)\}$. However, the function $\mathrm{ord}_A : A \to \mathbb{N} \cup \{\infty\}$ does not have this nice property in general. For example, let k be a field and let $A = (k[x, y]/(xy))_{(x,y)}$ (the local ring at the point at origin of the subscheme defined by $xy = 0$ in $\mathbb{A}^2_k$). Then $x, y \in Zerodiv(A)$ and $\mathrm{ord}_A(x) = \mathrm{ord}_A(y) = \infty$. But $x + y \in S(A)$ and $\mathrm{ord}(x + y) = 2$.

2.18 Exercise: Order is a homomorphism. (1) If A be a noetherian local ring of dimension 1, and $a \in S(A)$ (means a is a non zero-divisor in A), then show that the A-module A/aA is of finite length.
(2) With A, a as above, show that the sequence $0 \to A \xrightarrow{a} A \to A/(a) \to 0$ is exact, and so $\mathrm{ord}_A(a) = \ell(A/aA) = e_A(a, A)$.
(3) If $a, b \in S(A)$, then deduce using stmt 2.8 that

$$\mathrm{ord}_A(ab) = \mathrm{ord}_A(a) + \mathrm{ord}_A(b)$$

(4) Conclude that there is a unique well-defined group homomorphism

$$\mathrm{ord}_A : Q(A)^\times \to \mathbb{Z}$$

for extends $\mathrm{ord}_A : S(A) \to \mathbb{N}$.

2.19 Order for domains via finite birational ring extension. *Let A be a noetherian local domain of dimension 1, and let K be its quotient field. Let $A \subset B \subset K$ be another subring of K such that B is finite over A. Then the following holds.*

(1) For any $a \in A - \{0\}$, the A-modules A/aA and B/aB are of finite length, and moreover their lengths as A-modules are the same: $\ell(A/aA) = \ell_A(B/aB)$.

(2) If $\mathfrak{m} \subset A$ is the maximal ideal, and if $\mathfrak{n}_1, \ldots, \mathfrak{n}_r$ are the maximal ideals of B, then

$$\ell(A/aA) = \ell_A(B/aB) = \textstyle\sum_{1\le i\le r} \ell(B_{\mathfrak{n}_i}/aB_{\mathfrak{n}_i})[B_{\mathfrak{n}_i}/\mathfrak{n}_i B_{\mathfrak{n}_i} : A/\mathfrak{m}]$$

(3) If $r \in K^\times$ then $\mathrm{ord}_A(r) = \sum_{1\le i\le r} \mathrm{ord}\, B_{\mathfrak{n}_i}(r)[B_{\mathfrak{n}_i}/\mathfrak{n}_i B_{\mathfrak{n}_i} : A/\mathfrak{m}]$.

Proof of (1). As B is finite over A, the module B/A is finite over A. Moreover, being generically zero, B/A is set-theoretically supported over the closed point of $\mathrm{Spec}\, A$, so B/A is of finite length as an A-module. If a is a unit the statement is obvious, so let $a \in \mathfrak{m} - 0$. Then $\sqrt{(a)} = \mathfrak{m}$ as A is a 1-dimensional local domain, and so A/aA is of finite length being a quotient of $A/\mathfrak{m}^n$. Now the conclusion (1) follows by applying the statement **2.10** to the A-modules $A \subset B$, and noting that $a : A \to A$ and $a : B \to B$ are injective maps, so $e_A(a, A) = \ell(A/aA)$ and $e_A(a, B) = \ell_A(B/aB)$.

Proof of (2).

$$\begin{aligned}\ell_A(B/aB) &= \textstyle\sum_{1\le i\le r} \ell_A(B_{\mathfrak{n}_i}/aB_{\mathfrak{n}_i}) \text{ as } B/aB = \prod_{1\le i\le r} B_{\mathfrak{n}_i}/aB_{\mathfrak{n}_i},\\ &= \textstyle\sum_{1\le i\le r} \ell_{B_{\mathfrak{n}_i}}(B_{\mathfrak{n}_i}/aB_{\mathfrak{n}_i})[B_{\mathfrak{n}_i}/\mathfrak{n}_i B_{\mathfrak{n}_i} : A/\mathfrak{m}] \text{ by statement } \mathbf{2.13}.\end{aligned}$$

□

Proof of (3). We can write $r = a/a'$ where $a, a' \in A$ with $a' \neq 0$. As $\mathrm{ord}_A(a/a') = \ell(A/aA) - \ell(A/'aA)$ etc., applying (2) to a and a' and subtracting yields (3). □

2.20 Order via normalization. Let $(A, \mathfrak{m})$ be a 1-dim noetherian local domain. Suppose that the normalization B of A (means the integral closure of A inside its quotient field K) is finite over A. This implies that B is a noetherian ring that is *semi-local*, that is, it has only finitely many maximal ideals $\mathfrak{n}_1, \ldots, \mathfrak{n}_r$. Note that $\mathfrak{n}_i \cap A = \mathfrak{m}$. As B is normal, each $B_{\mathfrak{n}_i}$ is a discrete valuation ring. Let $\nu_i : B_{\mathfrak{n}_i} - \{0\} \to \mathbb{Z}$ denote the discrete valuation on $B_{\mathfrak{n}_i}$. Then for any $a \in A$ we have

$$\mathrm{ord}_A(a) = \textstyle\sum_{1\le i\le r} \nu_i(a)[B_{\mathfrak{n}_i}/\mathfrak{n}_i B_{\mathfrak{n}_i} : A/\mathfrak{m}].$$

The above is a special case of the stmt 2.19. In the examples of interest to us, the normalization B of A is indeed finite over A (in fact, a more general statement 2.21 holds).

2.21 Noether's theorem on finiteness of integral closure. Let A be an integral domain of essentially finite type (e.f.t.) over a field k, let K be the quotient field of A, let L/K be a finite extension of fields, and let $\tilde{A}$ be the integral closure of A in L. Then $\tilde{A}$ is finite over A and of e.f.t. over k. The proof uses Noether normalization as an essential ingredient, which is how the assumption of being e.f.t. over a field k is used.

Krull's theorem on finiteness of integral closure. Let $(A, \mathfrak{m})$ be a noetherian local domain of dimension 1, let K be its quotient field, and let $\tilde{A} \subset K$ be the normalization of A. The Krull-Akizuki theorem implies that $\tilde{A}$ is noetherian. However, $\tilde{A}$ need not be finite over A. In this matter, the noetherian condition on a ring is not as strong as the ring being e.f.t. over a base field. Krull's theorem says that $\tilde{A}$ is finite over A if and only if the completion of $\widehat{A}$ of A at $\mathfrak{m}$ is a reduced ring.

2.22 Lemma. (Needed for proving Proposition 2.24.) *Let A be a noetherian local domain of dimension 1. Suppose that the normalization of A in its quotient field is finite over A. Then for any finite free module $M = A^n$ over A and any A-linear endomorphism $\phi : M \to M$ which is generically an isomorphism, ϕ is of finite index and we have*

$$e_A(\phi, M) = \ell_A(\mathrm{coker}(\phi)) = \mathrm{ord}_A(\det(\phi)).$$

Proof. As a special case, suppose that A is normal, therefore a dvr. Let $\mathfrak{m} = (t)$ be its maximal ideal. Then M has two linear bases (u_i) and (v_i) such that $\phi(u_i) = t^{r_i}v_i$ where $r_i \geq 0$. It follows that $\mathrm{coker}(\phi) = \oplus_{1\leq i\leq n} A/(t^{r_i})$, so $\ell_A(\mathrm{coker}(\phi)) = \sum_i r_i$. Now, if $g \in GL_n(A)$ takes the basis v_i to the basis u_i, then $\det(g) \in A - \mathfrak{m}$, and ϕ has the matrix $g\, diag(t^{r_1}, \ldots, t^{r_n})$ w.r.t (u_i), so $\det(\phi) = \det(g) \cdot t^{r_1+\ldots+r_n}$. Hence $\mathrm{ord}_A(\det(\phi)) = \sum_i r_i$, showing that $\ell_A(\mathrm{coker}(\phi)) = \mathrm{ord}_A(\det(\phi))$ as claimed.

In the general case, let B be the normalization of A in its quotient field K. By assumption, B is finite over A. As B/A is a finite length A-module, from the exact sequence $0 \to A^n \to B^n \to (B/A)^n \to 0$ and the endomorphism $\phi \otimes 1_B : B^n \to B^n$ we see by statement **2.10** that

$$\ell_A(\mathrm{coker}(\phi)) = \ell_A(\mathrm{coker}(\phi \otimes 1_B)).$$

Let $\mathfrak{n}_1, \ldots, \mathfrak{n}_r$ be the primes in B lying above the maximal ideal $\mathfrak{m} \subset A$. These are exactly all the maximal ideals in B, and each $B_i = B_{\mathfrak{n}_i}$ is a dvr. Let $k_i = B_{\mathfrak{n}_i}/\mathfrak{n}_i B_{\mathfrak{n}_i}$ be its residue field. This is a finite extension of $k_A = A/\mathfrak{m}$. Consider $\phi \otimes 1_B : M \otimes_A B \to M \otimes_A B$. We have

$$\mathrm{coker}(\phi \otimes 1_B) = \oplus_i \,\mathrm{coker}(\phi \otimes 1_{B_i}).$$

Hence we get

$$\begin{aligned}
\ell_A(\mathrm{coker}(\phi)) &= \ell_A(\mathrm{coker}(\phi \otimes 1_B)) \\
&= \textstyle\sum_i \ell_A(\mathrm{coker}(\phi \otimes 1_{B_i})) \\
&= \textstyle\sum_i \ell_{B_i}(\mathrm{coker}(\phi \otimes 1_{B_i}))[k_i : k_A] \text{ by statement } \mathbf{2.13}, \\
&= \textstyle\sum_i \mathrm{ord}_{B_i}(\det(\phi \otimes 1_{B_i}))[k_i : k_A] \text{ as each } B_i \text{ is a dvr}, \\
&= \textstyle\sum_i \mathrm{ord}_{B_i}(\det(\phi))[k_i : k] \\
&= \mathrm{ord}_A(\det(\phi)) \text{ by } \mathbf{2.19}.(2).
\end{aligned}$$

$\square$

2.23 Comments on 2.19 and 2.24. In both these statements, we have a noetherian local domain A. The common assumption is that the normalization $\tilde{A}$ of A is finite over A.

In the statement 2.19, we take a finite extension $A \subset B \subset K$ where K is the quotient field of A (note that B need not be $\tilde{A}$, in fact, B need not even be normal). The statement tells us how to find $\mathrm{ord}_A(a)$ for $a \in A$ in terms of what happens over B. An important case is when $B = \tilde{A}$. This connects the definition of ord_A via lengths, to the older notion of order via discrete valuation rings. These dvr's are the localizations $B_{\mathfrak{n}_i}$, where $\mathfrak{n}_i$ are the maximal ideals of B (each of these lies over the maximal ideal $\mathfrak{m}$ of A).

The statement 2.24 begins with a finite extension of domains $A \subset B$ (which implies that the corresponding extension L/K of quotient fields is a finite extension). The statement 2.24 says that for any $r \in L^\times$, the direct image of the principal Weil divisor $[\mathrm{div}(r)]$ on $\mathrm{Spec}\, B$ under the finite morphism $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ is equal to the principal Weil divisor $[\mathrm{div}(Norm_{L/K}(r))]$ on $\mathrm{Spec}\, A$. This equality is crucial in the proof that proper direct images preserve rational equivalences.

2.24 Proposition : Finite direct image of a principal Weil divisor. *Let A be a 1-dimensional noetherian local domain, such that its integral closure in its quotient field K is finite over A. Let $A \subset B$ where B is a noetherian domain which is finite over A. Then we have the following.*

(1) The A-algebra $L = B \otimes_A K$, together with the induced A-algebra homomorphism $B \to L : b \mapsto b \otimes 1$, is the quotient field of B.

(2) The field extension L/K is finite, and under the norm map $\mathrm{Norm}_{L/K} : L \to K$, the image of B lies in A, inducing a multiplicative map $Norm_{L/K} : B \to A$.

(3) The ring B has only finitely many maximal ideals $\mathfrak{n}_1, \ldots, \mathfrak{n}_r$, where $r \geq 1$, and $\mathfrak{n}_i \cap A = \mathfrak{m}$ for each $1 \leq i \leq r$.

(4) For any $b \in B - \{0\}$ we have

$$\textstyle\sum_i \mathrm{ord}_{B_{\mathfrak{n}_i}}(b)[B_{\mathfrak{n}_i}/\mathfrak{n}_i B_{\mathfrak{n}_i} : A/\mathfrak{m}_A] = \ell_A(B/bB) = \mathrm{ord}_A(\mathrm{Norm}_{L/K}(b))$$

(5) If $r \in L^\times$ then

$$\textstyle\sum_i \mathrm{ord}_{B_{\mathfrak{n}_i}}(r)[B_{\mathfrak{n}_i}/\mathfrak{n}_i B_{\mathfrak{n}_i} : A/\mathfrak{m}_A] = \mathrm{ord}_A(\mathrm{Norm}_{L/K}(r))$$

Proof. The statements (1), (2), (3) are left to the reader.

Proof of (4): There exist elements $b_1, \ldots, b_n \in B$ such that $(b_i \otimes 1)$ is a K-linear basis for the vector space $L = B \otimes_A K$. Hence $bb_i \otimes 1 = \sum_j b_j \otimes c_{j,i}$ for some $c_{j,i} \in K$. Let $a \in A - \{0\}$ be a common denominator of all the $c_{j,i}$, so that $ac_{j,i} \in A$. The A-submodule $F \subset B$ generated by the b_i is free with basis (b_i). The A-linear map $ab : B \to B$ carries F into itself, as $(ab)c_{j,i} = \sum_j b_i(ac_{j,i})$. As the finite A-module B/F is supported over the closed point of $\mathrm{Spec}\, A$, it is of finite length, and we get

$$\begin{aligned} e_A(a, B) + e_A(b, B) &= e_A(ab, B) \text{ by statement } \mathbf{2.8}, \\ &= e_A(ab, F) \text{ by statement } \mathbf{2.10}, \end{aligned}$$

$$\begin{aligned}
&= \mathrm{ord}_A(\det(ab|F))) \text{ by statement } \mathbf{2.22}, \\
&= \mathrm{ord}_A(\mathrm{Norm}_{L/K}(ab)) \text{ as } (b_i) \text{ is a } K\text{-linear basis for } L, \\
&= \mathrm{ord}_A(\mathrm{Norm}_{L/K}(a)\,\mathrm{Norm}_{L/K}(b)) \text{ by } \mathbf{2.24.(2)}, \\
&= \mathrm{ord}_A(\mathrm{Norm}_{L/K}(a)) + \mathrm{ord}_A(\mathrm{Norm}_{L/K}(b)) \text{ by } \mathbf{2.18}.
\end{aligned}$$

This shows that $e_A(a,B)+e_A(b,B) = \mathrm{ord}_A(\mathrm{Norm}_{L/K}(a))+\mathrm{ord}_A(\mathrm{Norm}_{L/K}(b))$. Similarly, $e_A(a,B) = e_A(a,F) = \mathrm{ord}_A(\det(a|F)) = \mathrm{ord}_A(\mathrm{Norm}_{L/K}(a))$. From the last two equations, we get the equality $e_A(b,B) = \mathrm{ord}_A(\mathrm{Norm}_{L/K}(b))$. This establishes the second equality in 2.24.(4) as $e_A(b,B) = \ell_A(B/bB)$. The first equality in 2.24.(4) holds because $B/bB = \prod_i B_{\mathfrak{n}_i}/bB_{\mathfrak{n}_i}$, and by statement **2.13**, $\ell_A(B_{\mathfrak{n}_i}/bB_{\mathfrak{n}_i}) = \mathrm{ord}_{B_{\mathfrak{n}_i}}(b)[B_{\mathfrak{n}_i}/\mathfrak{n}_iB_{\mathfrak{n}_i} : A/\mathfrak{m}_A]$.
Proof of (5): Now express $r \in L^\times$ as b_1/b_2 for $b_1, b_2 \in B - \{0\}$. Applying the above to b_1 and b_2 and subtracting the results, we have

$$\begin{aligned}
&\textstyle\sum_i \mathrm{ord}_{B_{\mathfrak{n}_i}}(r)[B_{\mathfrak{n}_i}/\mathfrak{n}_iB_{\mathfrak{n}_i} : A/\mathfrak{m}_A] \\
=\;& \textstyle\sum_i \mathrm{ord}_{B_{\mathfrak{n}_i}}(b_1)[B_{\mathfrak{n}_i}/\mathfrak{n}_iB_{\mathfrak{n}_i} : A/\mathfrak{m}_A] - \sum_i \mathrm{ord}_{B_{\mathfrak{n}_i}}(b_2)[B_{\mathfrak{n}_i}/\mathfrak{n}_iB_{\mathfrak{n}_i} : A/\mathfrak{m}_A] \\
=\;& \mathrm{ord}_A(\mathrm{Norm}_{L/K}(b_1)) - \mathrm{ord}_A(\mathrm{Norm}_{L/K}(b_2)) \\
=\;& \mathrm{ord}_A(\mathrm{Norm}_{L/K}(r))
\end{aligned}$$

proving the result. □

2.25 Lemma used in 6.1 'push-pull equals pull-push' (which is left to the reader in [Fulton] Proposition 1.7). *Let L/K be a finite extension of fields, let A be an artin local K-algebra (not necessarily of finite type over K), let $B = A \otimes_K L$, let $\mathfrak{n}_1, \ldots, \mathfrak{n}_r$ be the maximal ideals of B, and let $B_i = B_{\mathfrak{n}_i}$. Then we have*

$$[L : K]\ell(A) = \textstyle\sum \ell(B_i)[B_i/\mathfrak{n}_iB_i : A/\mathfrak{m}_A]$$

Proof. Note that B is finite over A, so again an artin ring, but need not be local. As B is a free A-module of rank $[L : K]$, we have $\ell_A(B) = [L : K]\ell(A)$. By Chinese remainder theorem, $B = \prod B_i$. Note that the fundamental cycle of $\mathrm{Spec}\, B$ is $\sum \ell(B_i)[\mathfrak{n}_i]$, which is the flat pullback of the fundamental cycle of $\mathrm{Spec}\, L$. Its proper push forward to $\mathrm{Spec}\, A$ is by definition $\sum \ell(B_i)[B_i/\mathfrak{n}_iB_i : A/\mathfrak{m}_A][\mathfrak{m}_A]$. We will now show that $\ell_A(B) = \sum \ell_A(B_i) = \sum \ell(B_i)[B_i/\mathfrak{n}_iB_i : A/\mathfrak{m}_A]$. If $A \to C$ is a local homomorphism of artin local rings and M is a finite length C-module then $\ell_A(M) = \ell_C(M)[C/\mathfrak{m}_C : A/\mathfrak{m}_A]$. To see this, note that the graded terms in a composition series for the C-module M will all be $C/\mathfrak{m}_C$, and there will be $\ell_C(M)$ of these. Moreover, $\ell_A(C/\mathfrak{m}_C) = \ell_{A/\mathfrak{m}_A}(C/\mathfrak{m}_C) = [C/\mathfrak{m}_C : A/\mathfrak{m}_A]$. Applying this by taking $C = B_i$ gives the desired equality. Hence we get $[L : K]\ell(A) = \ell_A(B) = \sum \ell_A(B_i) = \sum \ell(B_i)[B_i/\mathfrak{n}_iB_i : A/\mathfrak{m}_A]$, which proves the lemma. □

2.26 [Fulton] Lemma A.2.7. *Let A be a 1-dimensional noetherian local ring, with maximal ideal $\mathfrak{m}$ and minimal primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$. Let M be a finite A-module, and let $a \in A - \cup_i \mathfrak{p}_i$. Then the index $e_A(a, M)$ is finite, given by the formula*

$$e_A(a,M) = \textstyle\sum_i \ell_{A_{\mathfrak{p}_i}}(M_{\mathfrak{p}_i})\, \ell\left(\dfrac{A}{\mathfrak{p}_i + (a)}\right) = \sum_i \ell_{A_{\mathfrak{p}_i}}(M_{\mathfrak{p}_i})\, \mathrm{ord}_{A/\mathfrak{p}_i}(a \bmod \mathfrak{p}_i).$$

Proof. Note that each $A_{\mathfrak{p}_i}$ is an artin local ring, and the localization $M_{\mathfrak{p}_i}$ is a finite module over it, so has a finite length. As $a \in A - \cup_i \mathfrak{p}_i$, the map $a : A/\mathfrak{p}_i \to A/\mathfrak{p}_i$ is injective for each $i = 1, \ldots, r$, and so we have $e_A(a, A/\mathfrak{p}_i) = \ell_A(A/(\mathfrak{p}_i + (a)))$. Hence we have to show that

$$e_A(a, M) = \textstyle\sum_i \ell_{A_{\mathfrak{p}_i}}(M_{\mathfrak{p}_i}) \cdot e_A(a, A/\mathfrak{p}_i).$$

By 1.15 the module M has a filtration $M = M_0 \supset M_1 \supset \ldots \supset M_n = 0$ where each graded piece is isomorphic to $A/\mathfrak{p}$ where $\mathfrak{p}$ a prime in A with $ann(M) \subset \mathfrak{p}$. As both sides of the above numerical equality are *additive* on M (that is, for a short exact sequence $0 \to M' \to M \to M'' \to 0$ the value on the middle term M is the sum of the values on the two terms M' and M''), it is enough to check for M of the form $A/\mathfrak{p}$ for all primes $\mathfrak{p} \subset A$. If $\mathfrak{p} = \mathfrak{m}$, then $\ell_A(M) = \ell_A(A/\mathfrak{m}) = 1 < \infty$, hence $e_A(a, M) = 0$. Also, $M_{\mathfrak{p}_i} = (A/\mathfrak{m})_{\mathfrak{p}_i} = 0$, so the other side is also zero. If $M = A/\mathfrak{p}_j$ for a minimal prime $\mathfrak{p}_j$, then $e_A(a, M) = e_A(a, A/\mathfrak{p}_j)$. On the other side, $M_{\mathfrak{p}_i} = (A/\mathfrak{p}_j)_{\mathfrak{p}_i}$ which equals 0 of $i \neq j$ and equals $A_j/\mathfrak{p}_j A_j$ if $i = j$ with $\ell_{\mathfrak{p}_j}(M_{\mathfrak{p}_j}) = 1$. Hence again we get the desired numerical equality. $\square$

2.27 Lemma: Commutativity and Serre's intersection index. *Let A be a noetherian local ring of dimension 2, with maximal ideal $\mathfrak{m}$. Let $a, b \in S(A)$ be non zero-divisor in A, such that either the ideal $(a, b) = (1)$ or the only prime ideal that contains (a, b) is $\mathfrak{m}$. Then we have the following.*

(1) The A-module $Tor_i^A(A/(a),\, A/(b))$ is of finite length for each $i \geq 0$, and $Tor_i^A(A/(a),\, A/(b)) = 0$ for each $i \geq 2$.

(2) Let $e = \sum_i (-1)^i \ell_A(Tor_i^A(A/(a),\, A/(b))) \in \mathbb{Z}$. Then

$$\sum_{\{\mathfrak{p} | a \in \mathfrak{p},\, ht(\mathfrak{p})=1\}} \mathrm{ord}_{A_\mathfrak{p}}(a)\, \ell\left(\frac{A}{\mathfrak{p} + (b)}\right) = e = \sum_{\{\mathfrak{p} | b \in \mathfrak{p},\, ht(\mathfrak{p})=1\}} \mathrm{ord}_{A_\mathfrak{p}}(b)\, \ell\left(\frac{A}{\mathfrak{p} + (a)}\right)$$

In particular, if $(a, b) = (1)$ then $e = 0$, and if $(a, b) \neq (1)$ then $e > 0$.

In geometric terms, if $a, b \in S(A) \cap \mathfrak{m}$ then a and b define principal effective Cartier divisors D and E in $X = \operatorname{Spec} A$, which are nonempty closed subschemes in X. The hypothesis that $\mathfrak{m}$ is the only prime ideal that contains a and b means that the intersection scheme $D \cap E$ is supported on the closed point $\mathfrak{m}$. Then the statement 2.27.(2) is equivalent to the equality

$$D \cdot [E] = E \cdot [D] = e[\mathfrak{m}]$$

where e is the alternating sum of lengths of the Tor modules, and moreover $e > 0$. This equality motivates the definition of the intersection class $D \cdot E$ of D and E, by putting $D \cdot E = e[\mathfrak{m}]$. Moreover, the equality $D \cdot [E] = E \cdot [D]$ is an important step in defining the intersection of Chow classes with an effective Cartier divisor.

Important! If $Tor_1^A(A/(a),\, A/(b)) \neq 0$ then $e \neq \ell(A/(a) \otimes_A A/(b)) = \ell(A/(a, b))$, and so the intersection class $D \cdot E$ is *not equal to the fundamental class* $[D \cap E] = \ell(A/(a, b))[\mathfrak{m}]$ *of the schematic intersection* $D \cap E$ of D and E in X.

Proof. If a or b is a unit (equivalently, if $(a, b) = (1)$), then the lemma is obvious, so let $a, b \in S(A) \cap \mathfrak{m}$. Then $A/(b)$ is a noetherian local ring of dimension 1, and the minimal prime ideals of $A/(b)$ are of the form $\mathfrak{p}/(b)$ where $\mathfrak{p}$ is a height 1 prime ideal in A such that $a \in \mathfrak{p}$. By assumption, $\overline{a} \in A/(b)$ does not lie any such $\mathfrak{p}/(b)$. Hence statement 2.26 ([Fulton] Lemma A.2.7.) applies to the ring $A/(b)$, the module $M = A/(b)$ over it, and its element $\overline{a} \in A/(b)$, to show that the index $e_A(\overline{a}, A/(b))$ is finite, and equals $\sum_{\{\mathfrak{p}|b\in\mathfrak{p},\, ht(\mathfrak{p})=1\}} \mathrm{ord}_{A_\mathfrak{p}}(b)\, \ell(A/(\mathfrak{p} + (a)))$. Similarly, $e_A(\overline{b}, A/(a))$ is finite, and equals $\sum_{\{\mathfrak{p}|a\in\mathfrak{p},\, ht(\mathfrak{p})=1\}} \mathrm{ord}_{A_\mathfrak{p}}(b)\, \ell(A/(\mathfrak{p} + (a)))$.

Now consider the A-module $M = A$ which has commuting endomorphisms a and b (acting by scalar multiplication). Both a and b are injective as $a, b \in S(A)$. We just saw that $\overline{a} : A/(b) \to A/(b)$ and $\overline{b} : A/(a) \to A/(a)$ are each of finite index, with the index given as above. Hence by Lemma 2.11, $e_A(\overline{a}, \mathrm{coker}(b)) = e_A(\overline{b}, \mathrm{coker}(a))$. This proves the equality $\sum_{\{\mathfrak{p}|b\in\mathfrak{p},\, ht(\mathfrak{p})=1\}} \mathrm{ord}_{A_\mathfrak{p}}(b)\, \ell(A/(\mathfrak{p} + (a))) = \sum_{\{\mathfrak{p}|a\in\mathfrak{p},\, ht(\mathfrak{p})=1\}} \mathrm{ord}_{A_\mathfrak{p}}(b)\, \ell(A/(\mathfrak{p} + (a)))$.

We have a free resolution $0 \to A \xrightarrow{a} A \to A/(a) \to 0$ of the A-module $A/(a)$. Tensoring with $A/(b)$ shows that $Tor_i^A(A/(a), A/(b))$ is the ith homology of the chain complex $0 \to A/(b) \xrightarrow{\overline{a}} A/(b) \to 0$. This proves (1), and moreover it shows that $e_A(\overline{a}, \mathrm{coker}(b)) = \sum_i (-1)^i\, \ell_A(Tor_i^A(A/(a),\, A/(b)))$. Interchanging a and b gives the equality $e_A(\overline{b}, \mathrm{coker}(a)) = \sum_i (-1)^i\, \ell_A(Tor_i^A(A/(a),\, A/(b)))$. Hence the result follows by [Fulton] Lemma A.2.8 (which is the statement 2.11 in these notes). □

2.28 Formula for $\mathrm{ord}_A(r)$ for $r \in Q(A)^\times$. *Let A be a 1-dimensional noetherian local ring, and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be its minimal prime ideals. Then for any $a \in S(A)$ a non zero-divisor in A, we have*

$$\ell(A/(a)) = \textstyle\sum_i \ell(A_{\mathfrak{p}_i})\, \ell\left(\dfrac{A}{\mathfrak{p}_i + (a)}\right).$$

In other words, if $a \mapsto a_i \in A/\mathfrak{p}_i$ under the quotient $A \to A/\mathfrak{p}_i$, then

$$\mathrm{ord}_A(a) = \textstyle\sum_i \ell(A_{\mathfrak{p}_i})\, \mathrm{ord}_{A/\mathfrak{p}_i}(a_i).$$

More generally, if $r \in Q(A)^\times$ has restrictions $r_i \in Q(A/\mathfrak{p}_i)^\times$ (if $r = a/b$ where $a, b \in S(A)$, then $a_i, b_i \in S(A/\mathfrak{p}_i)$ and $r_i = a_i/b_i \in Q(A/\mathfrak{p}_i)^\times$ which is well-defined), then

$$\mathrm{ord}_A(r) = \textstyle\sum_i \ell(A_{\mathfrak{p}_i})\, \mathrm{ord}_{A/\mathfrak{p}_i}(r_i).$$

Proof (see Proposition 3.(2) in [N]). By Proposition 2.16, $A/(a)$ is of finite length. In particular, its further quotients $A/(\mathfrak{p}_i + (a))$ are of finite length. As a is a non zero-divisor, $\ell(A/(a)) = e(a, A)$. Now the result for $\ell(A/(a))$ follows by applying the statement 2.26 to $M = A$. Finally for $r = a/b \in Q(A)$ where $a, b \in S(A)$, the desired formula for $\mathrm{ord}_A(r)$ follows by subtracting the formula for $\mathrm{ord}_A(b)$ from the formula for $\mathrm{ord}_A(a)$. □

3 Algebraic cycles and rational equivalence

Algebraic cycles

3.1 Algebraic schemes and morphisms. From here onwards in these notes, we will fix a base field k. This can be of any characteristic, and it need not be algebraically closed. An algebraic scheme over k means a scheme X together with a morphism of schemes $X \to \operatorname{Spec} k$ which is a morphism of finite type. By default a 'scheme' in what follows will mean an algebraic scheme over k, and a 'morphism' of schemes will be a morphism over k. An 'affine scheme' will mean an algebraic scheme isomorphic over k to $\operatorname{Spec} A$, where A is a finite type k-algebra. Note that every algebraic scheme X is noetherian, and so in particular the descriptions of $\mathcal{S}_X$, $\mathcal{K}_X$, $\mathcal{K}_X^\times$ etc for locally noetherian schemes apply to all algebraic schemes.

3.2 Varieties. Let X be a scheme (which now onwards means an algebraic scheme over k unless otherwise indicated). We say that X is a **variety** if it is integral (means reduced and irreducible). We **do not** require a variety be separated, unlike the use of the term 'variety' in most textbooks, in which separatedness is required. A **closed subvariety** of a scheme X is a closed subscheme $V \hookrightarrow X$ such that V is a variety.

3.3 Notation convention of [Fulton]. Let X be a scheme, and let $V \subset X$ be a closed subvariety of X. Let $\eta \in V$ be the generic point of V. Then following Fulton [F], we will use the notation $\mathcal{O}_{X,V}$ for the local ring $\mathcal{O}_{X,\eta}$ that is the stalk of $\mathcal{O}_X$ at η. This saves us from having to introduce a separate notation for generic points. Conversely, let $\eta \in X$ be a point, not necessarily closed. Let $V \subset X$ be the closure of $\{\eta\}$ in X, regarded as a reduced closed subscheme. Then V is a subvariety, and under this convention, $\mathcal{O}_{X,\eta}$ is denoted by $\mathcal{O}_{X,V}$.

3.4 The set $\mathfrak{V}_n(X)$ of all n-dimensional closed subvarieties of X. If X is any scheme, we denote by $\mathfrak{V}_n(X)$ the set of all n-dimensional closed subvarieties of X.

3.5 The subsets $X_{(n)} \subset X$ of points of transcendence degree n over k. For any n, let $X_{(n)} \subset X$ consist of all points $\eta \in X$ such that the residue field $\kappa(\eta) = \mathcal{O}_{X,\eta}/\mathfrak{m}_\eta$ satisfies $tr.deg_k(\kappa(\eta)) = n$, where for any field extension L/K we denote by $tr.deg_K(L)$ the transcendence degree of L over K.

3.6 Exercise: Identification of $\mathfrak{V}_n(X)$ and $X_{(n)}$. The residue field $\kappa(\eta)$ of the generic point $\eta \in V$ of any $V \in \mathfrak{V}_n(X)$ has transcendence degree $tr.deg_k(\kappa(\eta)) = n$ over k, which defines a map $\mathfrak{V}_n(X) \to X_{(n)}$. Conversely, if $\eta \in X_{(n)}$, let $V \subset X$ be the closure of $\{\eta\}$ in X with the reduced induced closed subscheme structure. Then $V \in \mathfrak{V}_n(X)$. This defines a map $X_{(n)} \to \mathfrak{V}_n(X)$. Show that the above maps $\mathfrak{V}_n(X) \to X_{(n)}$ and $X_{(n)} \to \mathfrak{V}_n(X)$ are bijections that are inverses of each other. We will identify $X_{(n)}$ with $\mathfrak{V}_n(X)$ under these bijections.

(2) If $U \subset U'$ are open subsets of X note that we have an inclusion $U_{(n)} \subset U'_{(n)}$. Define a map $\phi : \mathfrak{V}_n(U) \to \mathfrak{V}_n(U')$ which sends a closed subvariety V of U to its closure V' in U' (with reduced induced closed subscheme structure). Show that $V = V' \cap U$ as a scheme, and so the above map $\phi : \mathfrak{V}_n(U) \to \mathfrak{V}_n(U')$ is injective.

(3) Show that the following square commutes:

$$\begin{array}{ccc} U_{(n)} & \hookrightarrow & U'_{(n)} \\ \downarrow & & \downarrow \\ \mathfrak{V}_n(U) & \stackrel{\phi}{\to} & \mathfrak{V}_n(U') \end{array}$$

(4) Let X be a scheme, $Y \subset X$ a closed subscheme, and $U = X - Y \subset X$ the complementary open subscheme. Show that there are natural inclusions $Y_{(n)} \hookrightarrow X_{(n)}$ and $(X-Y)_{(n)} \hookrightarrow X_{(n)}$, and $X_{(n)}$ is the disjoint union

$$X_{(n)} = Y_{(n)} \coprod (X-Y)_{(n)}.$$

(5) Under the identifications given by (1) above, the inclusion $Y_{(n)} \hookrightarrow X_{(n)}$ corresponds to the natural inclusion $\mathfrak{V}_n(Y) \hookrightarrow \mathfrak{V}_n(X)$, under which an n-dimensional closed subvariety V of Y corresponds to itself regarded as an n-dimensional closed subvariety of X.

(6) Let X be a scheme, $Y \subset X$ a closed subscheme, and $U = X - Y \subset X$ the complementary open subscheme. Show that the natural inclusion $(X-Y)_{(n)} \hookrightarrow X_{(n)}$ corresponds to an injective map $\mathfrak{V}_n(X-Y) \hookrightarrow \mathfrak{V}_n(X)$ that sends $V \mapsto \overline{V}$, and the disjoint union $X_{(n)} = Y_{(n)} \coprod (X-Y)_{(n)}$ corresponds to a disjoint union $\mathfrak{V}_n(X) = \mathfrak{V}_n(Y) \coprod \mathfrak{V}_n(X-Y)$.

3.7 The group $Z_n(X)$ of n-cycles. Let X be a scheme, and let $n \in \mathbb{Z}$. The group $Z_n(X)$ of n-cycles on X is by definition the free abelian group over the set $\mathfrak{V}_n(X)$ of all n-dimensional closed subvarieties of X. Equivalently, $Z_n(X)$ is the free abelian group over the set $X_{(n)}$ of all points $\eta \in X$ with $tr.deg_k(\kappa(\eta)) = n$. If $V \in \mathfrak{V}_n(X)$, then the corresponding basis element of $Z_n(X)$ will be denoted by $[V]$. Note that $Z_n(X) = 0$ if $n < 0$ or if $n > \dim(X)$, and also $Z_n(\emptyset) = 0$. If $X \subset Y$ is a closed subscheme, then note that we have a natural injective homomorphism $Z_n(X) \to Z_n(Y)$ which sends $[V] \mapsto [V]$ where on the l.h.s. the symbol $[V]$ corresponds to a closed subvariety $V \hookrightarrow X$, and on the r.h.s. the symbol $[V]$ corresponds to the resulting closed subvariety $V \hookrightarrow X \hookrightarrow Y$.

3.8 The graded abelian group $Z_*(X)$. For any scheme X we define $Z_*(X) = \oplus_{n\in\mathbb{Z}} Z_n(X)$ with $\mathbb{Z}$-grading under which $Z_n(X)$ is the direct summand of degree n.

3.9 The presheaf $\mathcal{Z}_{X,n}$ of n-cycles. If $A \subset B$ are sets, and if $\mathbb{Z}A$ and $\mathbb{Z}B$ are the free abelian groups over these sets, then we have a natural projection $\mathbb{Z}B \to \mathbb{Z}A$, which maps the basis element $[b] \in \mathbb{Z}B$ that corresponds to $b \in B$ to 0 if $b \in B - A$ and to $[b] \in \mathbb{Z}A$ if $b \in A$. If $U \subset U'$ are open subschemes of X then for any n, we have $U_{(n)} \subset U'_{(n)}$, which induces a projection $Z_n(U') \to Z_n(U)$ where we identify $U_{(n)}$ with $\mathfrak{V}_n(U)$ and $U'_{(n)}$ with $\mathfrak{V}_n(U')$ following Exercise 3.6. Alternatively, the homomorphism $Z_n(U') \to Z_n(U)$ sends

any basis element $[V] \in Z_n(U')$, where $V \subset U'$ is a closed subvariety to the fundamental cycle $[V \cap U] \in Z_n(U)$ of $V \cap U$ in U (see 3.13 below) which is 0 if $V \cap U =$, and to the basis element $[V \cap U] \in Z_n(U)$ if $V \cap U \neq$.
Exercise. Using the Exercise 3.6, show that the association $U \mapsto Z_n(U)$ defines a presheaf on X in the Zariski topology.
We will denote presheaf $U \mapsto Z_n(U)$ on a scheme X by $\mathcal{Z}_{X,n}$. Consequently, the direct sum $\mathcal{Z}_{X,*} = \oplus_{n\in\mathbb{Z}} \mathcal{Z}_{X,n}$ is a graded presheaf in the Zariski topology on X.

3.10 Exercise: The presheaf $\mathcal{Z}_{X,n}$ is a sheaf of abelian groups in the Zariski topology on X (see [N] Proposition 2). Consequently, $\mathcal{Z}_{X,*}$ is a sheaf of graded abelian groups in the Zariski topology on X.

3.11 $\mathcal{Z}_{X,*}$ as a direct sum of skyscraper sheaves (see Rost [R]). Show that for any open subscheme $U \subset X$, we have an isomorphism of graded abelian groups

$$\oplus_{x\in U} \mathbb{Z}x \to Z_*(U)$$

where the left hand is the free abelian group generated by the underlying set of U, and the map sends $x \mapsto [\overline{x}]$ where $\overline{x} \subset U$ is the subscheme of U with reduced induced subscheme structure which is the closure $\overline{x}$ of $\{x\}$ in U, which is necessarily irreducible hence integral, making it a subvariety of U. The group $\oplus_{x\in U} \mathbb{Z}x$ is graded by defining a generator $x \in U$ to have degree n if $\dim(\overline{x}) = n$ (equivalently, the transcendence degree of the residue field extension $\kappa(x)/k$ is n). It is obvious from this alternative description that $\mathcal{Z}_{X,*}$ is a sheaf, for the isomorphisms $\oplus_{x\in U}\mathbb{Z}x \to Z_*(U)$ together induce an isomorphism of presheaves

$$\oplus_{x\in X}(i_x)_*\mathbb{Z}_x \to \mathcal{Z}_{X,*}$$

where $\mathbb{Z}_x$ denotes the constant sheaf $\mathbb{Z}$ on $\{x\}$, $i_x : \{x\} \hookrightarrow X$ denotes the inclusion map, and $(i_x)_*\mathbb{Z}_x$ denotes the corresponding direct image presheaf on X and $\oplus_{x\in X}$ is the direct image as presheaves as x varies over X. Now, the left hand side is already a sheaf in the Zariski topology on X, showing that $\mathcal{Z}_{X,*}$ is a sheaf.

3.12 Equidimensional schemes. We say that a scheme X is equidimensional of dimension d if each of its irreducible components is of dimension d. If X is nonempty and equidimensional of dimension d, then d is unique. On the other hand, the empty scheme $\emptyset$ is equidimensional of dimension d for each $d \in \mathbb{Z}$, as it has no irreducible components.

3.13 Fundamental cycle of a closed subscheme. Let Y be an equidimensional closed subscheme of dimension d of an algebraic scheme X. Let $Y_1, Y_2, \ldots, Y_n$ denote the irreducible components of Y, regarded as closed irreducible reduced subvarieties of X, where $n = 0$ if $Y = \emptyset$. Then by definition the fundamental cycle of Y is

$$[Y] = \sum \ell(\mathcal{O}_{Y,Y_i})[Y_i] \in Z_d(Y) \subset Z_d(X) \subset Z_*(X)$$

(see the notational convention 3.3). The length $\ell(\mathcal{O}_{Y,Y_i}$ is called the **geometric multiplicity** of Y_i in Y. In particular, $[\emptyset] = 0 \in Z_*(X)$.

Rational equivalence

3.14 The dimension of the support $|r|$ for $r \in \mathcal{K}^\times(X)$. Let $X \neq \emptyset$ be an equidimensional scheme of dimension d, and let $r \in \mathcal{K}^\times(X)$ be an invertible rational function on X. Let $|r|$ be the support of r as defined in statement 1.36. This is a closed subset of X, each of whose irreducible components has dimension $\leq d-1$. As X is noetherian, the closed subset $|r|$ is either empty or is a union of finitely many irreducible components. Hence X has at most finitely many closed subvarieties V of dimension $d-1$ such that $V \subset |r|$. If $\mathrm{ord}_V(r) \neq 0$ for a closed subvariety V of dimension $d-1$ then $V \subset |r|$ (Caution: Converse is not necessarily true). Hence X has at most finitely many closed subvarieties V of dimension $d-1$ such that $\mathrm{ord}_V(r) \neq 0$.

3.15 Weil divisor of a rational function on a variety. Let W variety and let $r \in R(W)^\times = \mathcal{K}^\times(W)$, a non-zero (equivalently, invertible) rational function on W. We define its Weil divisor $[r]$ (which is denoted by $[\mathrm{div}(r)]$ in [Fulton]) as an element of $Z_{\dim(W)-1}(W)$ by the following formula, where the sum is taken over all closed subvarieties $V \subset W$ such that $\dim(V) = \dim(W) - 1$ (which is a finite sum by statement 3.14).

$$[\mathrm{div}(r)] = [r] = \sum \mathrm{ord}_{\mathcal{O}_{W,V}}(r)\,[V] \in Z_{\dim(W)-1}(W)$$

3.16 The group homomorphism $\oplus_{W\in\mathfrak{V}_{d+1}(X)}\mathcal{K}^\times(W) \to Z_d(X)$. Let X be a scheme, and let $d \in \mathbb{Z}$. For any $W \in \mathfrak{V}_{d+1}(X)$ and $r \in \mathcal{K}^\times(W)$, we have defined above an element $[r] \in Z_d(X)$. By 2.18 applied to the 1-dimensional noetherian local ring $\mathcal{O}_{W,V}$, the map $\mathcal{K}^\times(W) \to Z_d(X) : r \mapsto [r]$ is a homomorphism of abelian groups. As W varies over the set $\mathfrak{V}_{d+1}(X)$ of all irreducible closed subvarieties of X of dimension $d+1$, by the universal property of coproducts (direct sums) of abelian groups, we get an induced homomorphism

$$\oplus_{W\in\mathfrak{V}_{d+1}(X)}\mathcal{K}^\times(W) \to Z_d(X).$$

3.17 Cycles rationally equivalent to zero. For any scheme X, the image of the above homomorphism $\oplus_{W\in\mathfrak{V}_{d+1}(X)}\mathcal{K}^\times(W) \to Z_d(X)$ is denoted by $Rat_d(X)$, which is a subgroup of $Z_d(X)$. Any two elements $\alpha, \beta \in Z_d(X)$ (that is, d-cycles on X) are said to be **rationally equivalent** if $\alpha - \beta \in Rat_d(X)$. Thus, $Rat_d(X)$ is the group of all d-cycles on X that are rationally equivalent to 0.

3.18 Exercise: The presheaf $U \mapsto Rat_d(U)$. (1) Show that $U \mapsto Rat_d(U)$ defines a sub presheaf $Rat_{X,d} \subset \mathcal{Z}_{X,d}$ of the sheaf $\mathcal{Z}_{X,d}$ of algebraic d-cycles on a scheme X.
(2) Show that when $X = \mathbb{P}^1$ the projective line over k, the presheaf $Rat_{X,0}$ is not separated: if $P \in X$ is a closed point, then the 0-cycle $[P]$ in $Z_0(X)$ is not in $Rat_0(X)$, but for the usual open cover $X = U_0 \cup U_\infty$ where U_0 and U_∞ are each isomorphic to $\mathbb{A}^1$, the restrictions of $[P]$ to U_0 and to U_∞ are each rationally equivalent to zero. In particular, $Rat_{X,d}$ is not a sheaf.

3.19 Definition of the Chow group $A_d(X) = Z_d(X)/Rat_d(X)$. The quotient group $A_d(X) = Z_d(X)/Rat_d(X)$ is called the d**-th Chow group of** X. The $\mathbb{Z}$-graded abelian group $A_*(X) = \oplus_{d\in\mathbb{Z}} A_d(X)$ is called the **total Chow group** of X.

3.20 Exercise: The presheaf $U \mapsto A_d(U)$. For any scheme X, note that we have a quotient presheaf $A_{X,d} = Z_{X,d}/Rat_{X,d}$ under which $U \mapsto A_d(U)$. Show that when $X = \mathbb{P}^1$, the presheaf $A_{X,d}$ is not a sheaf.

3.21 Exercise: The Chow group of $\operatorname{Spec} k$**.** Show that $Z_0(\operatorname{Spec} k)$ is the free abelian group on the single generator $[\operatorname{Spec} k]$, while $Z_i(\operatorname{Spec} k) = 0$ for $i \neq 0$. Deduce that $A_0(\operatorname{Spec} k) = \mathbb{Z}$ while $A_i(\operatorname{Spec} k) = 0$ for $i \neq 0$. More generally, show that the above statements hold $\operatorname{Spec} k$ replaced by any 0-dimensional variety over k (any such variety is isomorphic to $\operatorname{Spec} K$ where K is any finite extension field of k).

4 Proper push forward

4.1 The degree of a proper surjective morphism of varieties. Let $f : W \to V$ be a proper surjective morphism of varieties. Then f induces an extension of function fields $f^\sharp : R(f(W)) \hookrightarrow R(W)$. We now claim that the field extension $R(W)/R(V)$ is finite if and only if $\dim(W) = \dim(V)$. Note that $R(W)/R(V)$ is finitely generated as a field extension, as both are finitely generated field extensions of the base field k. Hence $R(W)/R(V)$ is finite if and only if it is algebraic. Now our desired conclusion follows the equalities $\dim(V) = tr.deg_k(R(V))$ and $\dim(W) = tr.deg_k(R(W))$.

Definition. The **degree** $\deg(f)$ of f, also called the degree of W over V and denoted by $\deg(W/V)$ if the morphism f is understood as given, is defined as follows.

$$\deg(f) = \deg(W/V) = \begin{cases} 0 & \text{if } \dim(V) < \dim(W), \text{ and} \\ [R(W) : R(V)] & \text{if } \dim(V) = \dim(W). \end{cases}$$

If $f : W \to V$ is a proper morphism of varieties which is not surjective, then we define $\deg(f) = 0$.

4.2 Exercise. (1) Let $f \in k[t]$ be a polynomial of degree d, and let $\mathbb{A}^1 = \operatorname{Spec} k[t]$. Show that the morphism $\phi : \mathbb{A}^1 \to \mathbb{A}^1$ induced by the k-algebra homomorphism $k[t] \to k[t] : t \mapsto f$ is proper and surjective if and only if $d \geq 1$, and in that case determine its degree.
(2) Let $\mathbb{P}^1 = \operatorname{Proj} k[x, y]$, and let $t = x/y \in R(\mathbb{P}^1)$. Show that t is transcendental over k, and the inclusion $k(t) \hookrightarrow R(\mathbb{P}^1)$ is an isomorphism over k. For any $r = f/g \in k(t) = R(\mathbb{P}^1)$, where $f, g \in k[t]$ are coprime and $g \neq 0$, let $\phi_r : \mathbb{P}^1 \to \mathbb{P}^1$ be the morphism that sends $(t, 1) \mapsto (f(t), g(t))$. What is $\deg(\phi_r)$ in terms $f, g \in k[t]$?

4.3 Proper push forward on algebraic cycles. Let $f : X \to Y$ be a proper morphism of schemes. For any $d \in \mathbb{Z}$, we associate to f a homomorphism $f_* : Z_d(X) \to$

$Z_d(Y)$, as follows. For any closed subvariety $V \subset X$ of dimension d, let $f(V)$ denote its image in Y regarded as a closed subvariety with the reduced induced subscheme structure. For the basis element $[V] \in Z_d(X)$, we define its image $f_*[V]$ in $Z_d(Y)$ by

$$f_*[V] = \deg(V/f(V))\,[f(V)].$$

4.4 Exercise: Functoriality of proper push forwards on algebraic cycles. Show the following.
(1) For any scheme X, the push forward corresponding to $\mathrm{id}_X : X \to X$ is $\mathrm{id}_{Z_*(X)} : Z_*(X) \to Z_*(X)$.
(2) If $f : X \to Y$ and $g : Y \to Z$ be proper morphisms of schemes then

$$(g \circ f)_* = g_* \circ f_* : Z_*(X) \to Z_*(Z).$$

4.5 Proper plus quasifinite is finite. Exercise: Any proper quasifinite morphism of schemes is a finite morphism.

4.6 Proposition: Push forward of a principal Weil divisor when rel dim = 0. *Let $f : X \to Y$ be a proper surjective morphism of varieties such that $\dim X = \dim Y$. Let $\mathrm{Norm}_{X/Y} : R(X) \to R(Y)$ denote the norm map for the corresponding finite extension of fields $f^\sharp : R(Y) \to R(X)$. Let $r \in R(X)^\times$, and let $[r]_X \in Z_{d-1}(X)$ be its Weil divisor. Then under $f_* : Z_{d-1}(X) \to Z_{d-1}(Y)$, we have*

$$f_*([r]_X) = [\mathrm{Norm}_{X/Y}(r)]_Y \in Z_{d-1}(Y)$$

where $\mathrm{Norm}_{X/Y}(r) \in R(Y)^\times$ and $[\mathrm{Norm}_{X/Y}(r)]_Y$ denotes its Weil divisor in $Z_{d-1}(Y)$.

Proof In the special case when f is finite, we have already proved this result: see Proposition 2.24. We now consider the more general case when f is proper of relative dimension 0, but not necessarily finite, and reduce the argument to the case of a related finite morphism of schemes over a new base field. These schemes may not be of finite type over the new base, but one is able to still apply Proposition 2.24.

Let $\dim X = \dim Y = d$. Both sides of the above equation are linear combinations of basis elements $[V] \in Z_{d-1}(Y)$ that correspond to closed subvarieties $V \subset Y$ of dimension $d-1$. Hence to prove the formula, we must check that the coefficient of any $[V]$ on the two sides of the equation is the same. So we now fix one such V. Let $y \in V$ be its generic point. By surjectivity of f, there exists a point $x \in X$ such that $f(x) = y$. We have an induced extension of residue fields $f^\sharp_x : \kappa(x) \to \kappa(y)$, which shows that $d - 1 = tr.deg_k(\kappa(y)) \leq tr.deg_k(\kappa(x))$. As $\dim(X) = d$, we must have $tr.deg_k(\kappa(x)) \leq d$. The only point z of X with $tr.deg_k(\kappa(z)) = d$ is the generic point of X. As f is surjective, the generic point of X maps to the generic point of Y, and so $x \neq z$. Hence $tr.deg_k(\kappa(x)) = d-1$. Hence x is the generic point of a closed subvariety W of X of dimension $d-1$, moreover, $f(W) = V$, and the field extension $f^\sharp_x : \kappa(y) \to \kappa(x)$ is finite. As the scheme $f^{-1}(V) \subset X$ is of dimension

$d-1$, it contains at most finitely many closed subvarieties W of dimension $d-1$. This shows that the set theoretic fiber of f over y is finite.

Let $A = \mathcal{O}_{Y,y}$ be the local ring of Y at y. This is a noetherian local domain of Krull dimension 1, which is e.f.t. over k, with quotient field $K = R(Y)$ and residue field $\kappa(y) = R(V)$. Because A is of e.f.t. over k, the normalization $\tilde{A}$ of A in K is finite over A by 2.21.

Let $Y' = \operatorname{Spec} \mathcal{O}_{Y,y}$ and let $f' : X' \to Y'$ be the base change of f under the inclusion morphism $Y' \hookrightarrow Y$ (warning: the schemes X' and Y' are not of finite type over k). The only points of Y' are y_0 and y where y_0 is the generic point of Y, and y is the point that we began with, which is the unique closed point of Y'. The set-theoretic fiber of $f : X \to Y$ over $y_0 \in Y$ is a singleton set, consisting of the generic point $x_0 \in X$, as seen by a transcendence degree argument similar to the one made above. As already seen, the set-theoretic fiber of $f : X \to Y$ over $y \in Y$ is a finite set. Also, with notation as above, the induced field extensions $R(Y) \hookrightarrow R(X)$ and $R(V) \hookrightarrow R(W)$ are finite. This shows that the proper morphism $f' : X' \to Y'$ is quasifinite, and hence finite (see 4.5). Hence $X' = \operatorname{Spec} B$ for a finite A-algebra B.

We now show that X' is an integral scheme, and consequently, B is an integral domain. Let $\operatorname{Spec} C$ be an affine open neighbourhood of $y \in Y$ and let $\operatorname{Spec} D$ be an affine open neighbourhood of $x \in X$ such that $f(\operatorname{Spec} D) \subset \operatorname{Spec} C$. Note that both C and D are domains of finite type over k. Let $\mathfrak{p} \subset C$ be the prime ideal corresponding to y. Then $D_{\mathfrak{p}} = D \otimes_C C_{\mathfrak{p}}$ is again a domain, and $\operatorname{Spec} D_{\mathfrak{p}} \subset X'$ is an affine open neighbourhood of $x \in X'$. Also note that $x_0 \in \operatorname{Spec} D_{\mathfrak{p}}$. As $D_{\mathfrak{p}}$ is the localization of a domain D, it is a domain, and so X' is locally an integral scheme. As x_0 is dense in X', the scheme X' is irreducible. Being irreducible and locally integral, it is integral.

Under $Y' \hookrightarrow Y$, the generic point of Y' maps to the generic point of Y, and under $X' \hookrightarrow X$, the generic point of X' maps to the generic point of X, inducing identifications $R(X) = R(X')$ and $R(Y) = R(Y')$.

The closed point $y \in Y'$ corresponds to the maximal ideal $\mathfrak{m} \subset A$. The closed points of B are defined by its maximal ideals $\mathfrak{n}_1, \ldots \mathfrak{n}_r$ where $r \geq 1$, these contract to $\mathfrak{m}$, and and they correspond to the generic points of closed subvarieties $W \subset X$ of dimension $d-1$ such that $f(W) = V$.

With all this preparation, we now show that $f_*[div_X(r)] = [div_Y(\operatorname{Norm}(r))]$, by checking that the numerical coefficient of $[V]$ on the two sides of the equation is the same. By definition the numerical coefficient of $[V]$ in $[div_Y(\operatorname{Norm}(r))] \in Z_{d-1}$ is $\operatorname{ord}_V(\operatorname{Norm}(r)) = \operatorname{ord}_A(\operatorname{Norm}(r))$. On the other hand, the numerical coefficient of $[V]$ in $f_*[div_X(r)]$ is defined as follows. By definition, $[div_X(r)] = \sum_{W \in \mathfrak{V}_{d-1}(X)} \operatorname{ord}_W(r)\,[W]$. If $f(W)$ has dimension $d-1$, then $f_*[W] = \deg(W/V)[f(W)]$ where $\deg(W/V) = [R(W) : R(f(W))] \geq 1$. If $f(W)$ has dimension $< d-1$ then $f_*([W]) = 0$. Hence for our given $[V]$, the numerical coefficient of $[V]$ in $f_*[div_X(r)]$ is

$$\sum_{\{W \in \mathfrak{V}_{d-1}(X) \,|\, f(W) = V\}} \operatorname{ord}_W(r)\,[R(W) : R(V)] = \sum_i \operatorname{ord}_{B_{\mathfrak{n}_i}}(r)[B_{\mathfrak{n}_i}/\mathfrak{n}_i B_{\mathfrak{n}_i} : A/\mathfrak{m}_A].$$

By equation 2.24.(5), we have

$$\sum_i \mathrm{ord}_{B_{\mathfrak{n}_i}}(r)[B_{\mathfrak{n}_i}/\mathfrak{n}_i B_{\mathfrak{n}_i} : A/\mathfrak{m}_A] = \mathrm{ord}_A(\mathrm{Norm}_{L/K}(r)).$$

Hence the result follows. $\square$

4.7 Theorem (Chevalley). **Proper push forwards preserve rational equivalence.** *Let $f : X \to Y$ be a proper morphism of schemes. If $\alpha \in Z_d(X)$ is a d-cycle on X (where $d \geq 0$) that is rationally equivalent to 0 on X, then its push forward $f_*(\alpha) \in Z_d(Y)$ is rationally equivalent to 0 on Y, in other words, $f_*(Rat_d(X)) \subset Rat_d(Y)$. Thus, there is an induced homomorphism $f_* : A_d(X) \to A_d(Y)$, making each A_d a covariant functor from the category of schemes and proper morphisms to the category of abelian groups.*

Proof. Any such α is a finite linear combination of d-cycles on X of the form $[r_i]$ where r_i is a nonzero rational function on a $d+1$-dimensional closed subvariety W_i of X. Hence to prove that $f_*\alpha$ is rationally equivalent to 0 in Y, it is enough to show that if $W \subset X$ is a closed subvariety of dimension $d+1$ and $r \in R(W)^\times$, then $f_*[r]$ is rationally equivalent to 0 in Y. For this we make the following three cases, where V denotes the closed subvariety $f(W)$ of Y.

Case A: $\dim(W) \geq \dim(V) + 2$. For $r \in R(W)^\times$, let $[r] = \sum_{W' \in \mathfrak{V}_d(W)} \mathrm{ord}_{W'}(r)[W'] = \sum_j n_j W'_j$. Then $\dim f(W'_j) \leq \dim(V) \leq \dim(W) - 2 = \dim(W'_j) - 1$, showing that $\dim f(W'_j) < \dim(W'_j) = d$. Hence by definition of $f_* : Z_d(X) \to Z_d(Y)$, we have $f_*[W'_j] = 0 \in Z_d(Y)$. Hence $f_*[r] = 0 \in Z_d(Y)$. This conclusion is stronger than the cycle $f_*[r]$ merely being rationally equivalent to 0 in Y.

Case B: $\dim(W) = \dim(V)$. This is just the Proposition 4.6, which we have already proved.

Case C: $\dim(W) = \dim(V)+1$. We treat this case using the Theorem 4.11 that is proved below. The proof of the Theorem 4.11 uses the Proposition 4.6 in a simpler situation, where the proper morphism is in fact finite, which we can directly treat using equation 2.24.(5) as was done in the proof of the Proposition 4.6.

Let $\mathrm{Spec}\, R(V) \hookrightarrow V$ be the generic point of V, and let $C \to \mathrm{Spec}\, R(V)$ denote the base-change of $W \to V$. Then C is a complete curve over the field $R(V)$. The inclusion $C \hookrightarrow W$ sends the generic point of C to the generic point of W, and it induces an isomorphism $R(W) \to R(C)$, hence the given rational function $r \in R(W)^\times$ defines a rational function $r_C \in R(C)^\times$. Let $[r] = \sum_{W' \in \mathfrak{V}_d(W)} \mathrm{ord}_{W'}(r)[W'] = \sum_j n_j W'_j$. For each $W' \in \mathfrak{V}_d(W)$, there are two possibilities: either $W' \to V$ is dominant, or $W' \to V$ is not dominant, which exactly correspond to whether the generic point of W' maps to the generic point of V or not. This means that the closed points of C are the same as the generic points of those $W' \in \mathfrak{V}_d(W)$ for which $W' \to V$ is dominant. If $W' \to V$ is not dominant, then $\dim(W') = d = \dim(V) > \dim(f(W'))$. Hence by definition of f_* on cycles, we have $f_*[W'] = 0$. $W' \to V$ is dominant, equivalently, the generic point of W' is a closed point x of C, then $R(W')$ is the residue field of C at x. Moreover, $\mathrm{ord}_{W'}(r) = \mathrm{ord}_x(r_C)$. Hence we have

$$f_*[r] \;=\; \sum_{W' \in \mathfrak{V}_d(W) \text{ such that } f(W')=V} \mathrm{ord}_{W'}(r) f_*[W']$$

$$\begin{aligned}
&= \textstyle\sum_{W'\in\mathfrak{V}_d(W) \text{ such that } f(W')=V} \mathrm{ord}_{W'}(r)[R(W'):R(V)][V] \\
&= \textstyle\sum_{x\in\mathfrak{V}_0(C)} \mathrm{ord}_x(r_c)[R(x):R(V)][V] \\
&= \deg(r_C)[V] \text{ by the Definition 4.8 of the degree of a 0-cycle on } C \\
&= 0 \text{ by Theorem 4.11.}
\end{aligned}$$

This completes the proof of the Case C modulo the proof of Theorem 4.11. □

4.8 Definition: The degree of a 0-cycle on a scheme. Let X be a scheme. If $P \in X_{(0)}$ is a closed point of X, the field extension $\kappa(P)/k$ is finite. The definition 4.1 of degree of a proper morphism when applied to the projection $\mathrm{Spec}\,\kappa(P) \to \mathrm{Spec}\,k$ gives $\deg(P/\mathrm{Spec}\,k) = [\kappa(P) : k]$. We simply write it as $\deg(P)$, with k understood as given. Let $\alpha \in Z_0(X)$ be a 0-cycle. By definition, α is uniquely expressible as a finite linear combination $\sum_i m_i[P_i]$ where $P_i \in X_{(0)}$ are closed points. We define the **degree of the 0-cycle** α by

$$\deg(\alpha) = \sum m_i \deg(P_i).$$

If the structure morphism $p : X \to \mathrm{Spec}\,k$ is proper, then by definition of the push forward, for any $P \in X_{(0)}$ we have $p_*[P] = \deg(P)[\mathrm{Spec}\,k] \in Z_0(\mathrm{Spec}\,k)$ and so

$$p_*\alpha = \deg(\alpha)[\mathrm{Spec}\,k] \in Z_0(\mathrm{Spec}\,k).$$

4.9 Exercise. (1) Let k be any field and $p : C \to \mathrm{Spec}\,k$ a proper morphism of schemes where C is irreducible, reduced, of dimension 1. Show that the normalization $\pi : \tilde{C} \to C$ of C is finite over C, the composite $p \circ \pi : \tilde{C} \to \mathrm{Spec}\,k$ is proper, and the scheme $\tilde{C}$ is 1-dimensional, irreducible and regular. The induced map of fields $\pi^\sharp : R(C) \to R(\tilde{C})$ is an isomorphism. If $f \in R(\tilde{C})^\times$ is transcendental over k, then there exists a unique finite surjective morphism of k-schemes $\phi : \tilde{C} \to \mathbb{P}^1_k = \mathrm{Proj}\,k[x,y]$ such that under the corresponding homomorphism of fields $\phi^\sharp : R(\mathbb{P}^1_k) \to R(\tilde{C})$ over k, we have $y/x \mapsto f$.

(2) (Less elementary. So though it can shorten the proof of Theorem 4.11, we will make do with the statement (1) which is weaker.) Let k be any field and $p : C \to \mathrm{Spec}\,k$ a proper morphism of schemes where C is irreducible, reduced, of dimension 1. Show that there exists a finite surjective morphism of k-schemes $\phi : C \to \mathbb{P}^1_k$.

4.10 Exercise. Any principal Weil divisor on $\mathbb{P}^1_k$ has degree 0.

4.11 Theorem: Any principal Weil divisor on a complete curve has degree 0. *Let $p : C \to \mathrm{Spec}\,k$ be a proper variety of dimension 1 over a base field k. Let $r \in R(C)^\times$ be a non-zero rational function on C, and let $[r] = [\mathrm{div}(r)] \in Z_0(C)$ be its Weil divisor. Then $\deg([r]) = 0$, equivalently, $p_*[r] = 0 \in Z_0(\mathrm{Spec}\,k)$.*

Proof Let $\pi : \tilde{C} \to C$ be the normalization of C. This enjoys the properties given by Exercise 4.9.(1) above. As $R(\tilde{C})$ has transcendence degree 1 over k, there exists $f \in R(\tilde{C})^\times$ which is transcendental over k. Let $\phi : \tilde{C} \to \mathbb{P}^1_k$ be the corresponding finite morphism as given by Exercise 4.9.(1). Let $q : \mathbb{P}^1_k \to \mathrm{Spec}\, k$ be the structure morphism. Then $q \circ \phi = p \circ \pi : \tilde{C} \to \mathrm{Spec}\, k$. Under the isomorphism $\pi^\sharp : R(C) \to R(\tilde{C})$ over k, let $r \mapsto \tilde{r}$. Then $Norm_{R(\tilde{C})/R(C)}\, \tilde{r} = r$. Hence $\pi_*([\tilde{r}]_{\tilde{C}}) = [r]_C$. This shows that $p_*[r] = p_*\pi_*([\tilde{r}]_{\tilde{C}}) = q_* \circ \phi_*([\tilde{r}]_{\tilde{C}}) = q_*([Norm_{R(\tilde{C})/R(\mathbb{P}^1)}\, \tilde{r}]_{\mathbb{P}^1_k})$. By Exercise 4.10, it follows that $q_*([Norm_{R(\tilde{C})/R(\mathbb{P}^1)}\, \tilde{r}]_{\mathbb{P}^1_k})$ equals $0 \in Z_0(\mathrm{Spec}\, k)$. □

5 Flat pullback

5.1 Flatness and dimension of fibers (see Hartshorne [H] III.9.5). Let $f : X \to Y$ be a flat morphism of finite type, where X and Y are noetherian. Then for any $x \in X$, the dimension at x of the fiber $X_{f(x)} = f^{-1}f(x)$ is given by

$$\dim_x(X_{f(x)}) = \dim_x(X) - \dim_{f(x)}(Y).$$

5.2 Definition: Morphisms of relative dimension n. If $f : X \to Y$ is a finite-type morphism of noetherian schemes and $n \geq 0$ is an integer, then we say that f has relative dimension n if every fiber of f is equidimensional of dimension n if non-empty (where **equidimensionality of a scheme** means that its every irreducible component has the same dimension).

5.3 Flat morphism of constant relative dimension n (see Hartshorne [H] III.9.6). Let X and Y be non-empty schemes of finite type over a field k, such that Y is irreducible. Let $f : X \to Y$ be a flat k-morphism. Then f is of relative dimension n if and only if X is equidimensional of dimension equal to $\dim(Y) + n$.

5.4 Exercise: Base change If $f : X \to Y$ is a flat morphism of constant relative dimension n and $Y' \to Y$ is any morphism, then the base change $f' : X' \to Y'$ (where $X' = X \times_Y Y'$ and f' is the projection) is again a flat morphism of constant relative dimension n.

5.5 Pull back on cycles under a flat morphism of relative dimension n. Let $f : X \to Y$ be a flat morphism of relative dimension n of algebraic schemes. Then we define $f^* : Z_dY \to Z_{d+n}X$ by $[V] \mapsto [f^{-1}V]$ for any closed subvariety $V \subset Y$ of dimension d, where $[f^{-1}V]$ is the fundamental cycle of the schematic inverse image $f^{-1}V$ of V.

5.6 Proposition: Pull-back for fundamental cycles. Let $f : X \to Y$ be a flat morphism of relative dimension n of algebraic schemes. If $Y' \subset Y$ is an equidimensional closed subscheme of dimension d, then we have

$$[f^{-1}Y'] = f^*[Y'] \in Z_{d+n}X$$

Proof. We apply the equality $\ell_B(B \otimes_A M) = \ell_A(M)\ell_B(B/\mathfrak{m}_A B)$ (see statement **2.14**), by taking B to be the local ring $\mathcal{O}_{f^{-1}Y',W}$ where W is an irreducible component of Y', taking A to be the local ring $\mathcal{O}_{Y',V}$ where $V = \overline{f(W)}$ is closure of the image $f(W)$, and taking $M = A$. This gives $\ell_B(B) = \ell_A(A)\ell_B(B/\mathfrak{m}_A B)$. Note that $\ell_B(B)$ is the coefficient of $[W]$ in $[f^{-1}Y'] \in Z_{d+n}X$, and $\ell_A(A)$ is the coefficient of $[V]$ in $[Y'] \in Z_dY$. As by definition $f^*[V] = [f^{-1}V]$, and as the local ring $\mathcal{O}_{f^{-1}V,W}$ equals $B/\mathfrak{m}_A B$, we see that $\ell_B(B/\mathfrak{m}_A B)$ is the coefficient of $[W]$ in $f^*[V] \in Z_{d+n}X$. Therefore the equality $\ell_B(B \otimes_A M) = \ell_A(M)\ell_B(B/\mathfrak{m}_A B)$ says that the coefficients of $[W]$ in $[f^{-1}Y'] \in Z_{d+n}X$ and in $f^*[Y'] \in Z_{d+n}X$ are the same. $\square$

5.7 Functoriality of flat pullbacks. Let $f : X \to Y$ and $g : Y \to Z$ be flat morphisms of algebraic schemes, of relative dimension m and n respectively. Then $g \circ f : X \to Z$ is flat of relative dimension $m + n$, and

$$(g \circ f)^* = f^* \circ g^* : Z_dZ \to Z_{d+m+n}X.$$

This equality is an immediate consequence of statement **5.6**.

5.8 Proposition. (See [N], Proposition 3.)
Let a scheme X have pure dimension n, and let $X_1, \ldots, X_p$ be the irreducible components of X, each regarded as a closed subvariety of X of dimension n. Then the following statements hold.

(1) For any $s \in \mathcal{S}(X)$, we have $[s]_X = [X/(s)] \in Z_{n-1}(X)$ where $[X/(s)]$ is the fundamental cycle of the closed subscheme $X/(s) \subset X$ defined by the ideal sheaf $(s) \subset \mathcal{O}_X$.

(2) Let $r \in \mathcal{K}^\times(X)$, and let $r_i \in \mathcal{K}^\times(X_i)$ be its restriction to X_i (see statement 1.37). Then we have $[r]_X = \sum_i \ell(\mathcal{O}_{X,X_i})\,[r_i]_{X_i} \in Rat_{n-1}(X) \subset Z_{n-1}(X)$.

Proof. (1) If $s \in \mathcal{S}(X)$ then the support $|s|$ of s (see statement 1.36 for the definition of $|s|$) is the underlying closed subset $|X/(s)|$ of the closed subscheme $X/(s)$ of X. Let V_j be the irreducible components of $X/(s)$. By definition, $[s]_X = \sum_{V_j} \mathrm{ord}_{\mathcal{O}_{X,V_j}}(s)\,[V_j] = \sum_{V_j} \ell(\mathcal{O}_{X,V_j}/(s))\,[V_j]$. On the other hand, the fundamental cycle $[X/(s)]$ of the scheme $X/(s)$ is given by $[X/(s)] = \sum_{V_j} \ell(\mathcal{O}_{X/(s),V_j})\,[V_j]$. As $\mathcal{O}_{X,V_j}/(s) \cong \mathcal{O}_{X/(s),V_j}$, the numerical coefficients of each $[V_j]$ on the two sides are equal, hence (1) holds.

(2) Now let $r \in \mathcal{K}^\times(X)$. Let $X_1, \ldots, X_p$ (where $p \geq 0$) be the irreducible components of X, each regarded as a closed subvariety of X of dimension n. As the support $|r|$ does not contain any height 0 point (see statement 1.36), there are only finitely many closed subvarieties $V_1, \ldots, V_q$ of dimension $n-1$ of X (where $q \geq 0$) that are contained in $|r|$. If V is a closed subvariety of X of dimension $n-1$ whose generic point does not lie in $|r|$ then $\mathrm{ord}_V(r) = 0$. Hence $[r]_X = \sum_{j=1}^q \mathrm{ord}_{\mathcal{O}_{X,V_j}}(r)[V_j]$ (however, it is possible that $\mathrm{ord}_{\mathcal{O}_{X,V_j}}(r) = 0$ for some V_j even though $V_j \subset |r|$).

By definition $[r_i]_{X_i} = \sum_{\{i \,|\, V_j \subset X_i\}} \mathrm{ord}_{\mathcal{O}_{X_i,V_j}}(r_i)\,[V_j]$. Hence $[r]_X = \sum_i \ell(\mathcal{O}_{X,X_i})\,[r_i]_{X_i}$ if and only if

$$\textstyle\sum_j \mathrm{ord}_{\mathcal{O}_{X,V_j}}(r)[V_j] = \sum_{\{(i,j) \,|\, V_j \subset X_i\}} \ell(\mathcal{O}_{X,X_i})\, \mathrm{ord}_{\mathcal{O}_{X_i,V_j}}(r_i)\,[V_j].$$

Comparing the numerical coefficients of $[V_j]$ on both sides, the above holds if and only if for each j we have

$$\mathrm{ord}_{\mathcal{O}_{X,V_j}}(r) = \sum_{\{i \mid V_j \subset X_i\}} \ell(\mathcal{O}_{X,X_i})\, \mathrm{ord}_{\mathcal{O}_{X_i,V_j}}(r_i). \qquad \ldots(3)$$

Now note that for any V_j, the ring $\mathcal{O}_{X,V_j}$ is a noetherian local ring of dimension 1, whose minimal prime ideals exactly correspond to the irreducible components X_i which contain V_j. The equality (3) follows now from the statement 2.28 applied to the 1-dimensional noetherian local ring $A_j = \mathcal{O}_{X,V_j}$ for each V_j.

As each r_i is a non-zero rational function on the variety X_i, by definition $[r_i]_{X_i}$ belongs to $Rat_{n-1}(X)$, and hence $[r]_X \in Rat_{n-1}(X)$ as desired. □

5.9 Proposition. *Let X and Y be equidimensional schemes of dimensions $n+d$ and n respectively, and let $f : X \to Y$ be a flat morphism of constant relative dimension d. Then the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{K}^\times(Y) & \stackrel{f^\sharp}{\to} & \mathcal{K}^\times(X) \\ \downarrow & & \downarrow \\ Z_{n-1}(Y) & \stackrel{f^*}{\to} & Z_{n+d-1}(X) \end{array}$$

where the vertical maps associate to any invertible meromorphic function its corresponding Weil divisor. In other words, if $r \in \mathcal{K}^\times(Y)$ pulls back to $f^\sharp r \in \mathcal{K}^\times(X)$ under f, and if $[r]_Y \in Z_{n-1}(Y)$ and $[f^\sharp r]_X \in Z_{n+d-1}(X)$ are the respective Weil divisors, then

$$f^*([r]_Y) = [f^\sharp r]_X \in Z_{n+d-1}(X).$$

Proof Case 1 - Open embeddings: Let $f : X \to Y$ be an open embedding. In this case the equality $f^*[r] = [f^\sharp r]$ becomes $[r]|_X = [r|_X]$, which is immediate from the definitions.

Case 2 - Affine case: Let $X = \mathrm{Spec}\, A$, $Y = \mathrm{Spec}\, B$. Note that f is induced by the flat k-algebra homomorphism $f^\sharp : B \to A$. Then $\mathcal{K}^\times(X) = Q(A)^\times$ and $\mathcal{K}^\times(Y) = Q(B)^\times$, and the flat homomorphism $f^\sharp : B \to A$ sends $S(B)$ into $S(A)$, thus inducing a homomorphism $f^\sharp : Q(B) \to Q(A)$. Hence $r = b_1/b_2$ where $b_1, b_2 \in S(B)$. Then $[r] = [b_1] - [b_2]$, and so $f^*[r] = f^*[b_1] - f^*[b_2]$. On the other hand, $f^\sharp r = (f^\sharp b_1)/(f^\sharp b_2)$, and so $[f^\sharp r] = [f^\sharp b_1] - [f^\sharp b_2]$. Hence, to prove that $f^*[r] = [f^\sharp r]$, it is enough to prove that $f^*[b] = [f^\sharp b]$ for all $b \in S(B)$.

As $b \in S(B) = \mathcal{S}(Y)$, by Proposition 5.8.(1) we have $[b]_Y = [\mathrm{Spec}\, B/(b)] \in Z_{n-1}(\mathrm{Spec}\, B)$, and so $f^*[b] = f^*[\mathrm{Spec}\, B/(b)]$. By definition of flat pullback on cycles, $f^*[\mathrm{Spec}\, B/(b)] = [f^{-1}(\mathrm{Spec}\, B/(b))]$, which is the fundamental cycle of the closed subscheme $f^{-1}(\mathrm{Spec}\, B/(b))$ of $\mathrm{Spec}\, A$. The schematic inverse image $f^{-1}(\mathrm{Spec}\, B/(b))$ is the closed subscheme $\mathrm{Spec}\, A/(f^\sharp b)$ of $\mathrm{Spec}\, A$. The element $f^\sharp b$ lies in $S(A) = \mathcal{S}(X)$ as $f^\sharp : B \to A$ is flat. Hence by Proposition 5.8.(1) we have $[f^\sharp b] = [\mathrm{Spec}\, A/(f^\sharp b)]$. Hence we have the sequence of equalities

$$f^*[b] = f^*[\mathrm{Spec}\, B/(b)] = [f^{-1}(\mathrm{Spec}\, B/(b))] = [\mathrm{Spec}\, A/(f^\sharp b)] = [f^\sharp b]$$

in $Z_{n+d-1}(\mathrm{Spec}\, A)$.

Case 3 - General case: (It is useful to refer to diagram 5.10 below while reading the proof.) Recall from 1.32 and 3.10 the definitions of the Zariski sheaves $\mathcal{K}_X^\times$ and $\mathcal{Z}_{X,n+d-1}$ on X. We have two global sections $f^*[r]$, $[f^\sharp r] \in Z_{n+d-1}(X) = \Gamma(X,\, \mathcal{Z}_{X,n+d-1})$ of the sheaf $\mathcal{Z}_{X,n+d-1}$, and we want to show that they are equal. Hence it is enough to show that their restrictions to each open set U_i are equal for some open cover (U_i) of X.

Let $\operatorname{Spec} B$ be an affine open subscheme of Y, and let $\operatorname{Spec} A$ be an affine open subscheme of X such that $\operatorname{Spec} A \subset f^{-1} \operatorname{Spec} B$. Let $f' : \operatorname{Spec} A \to \operatorname{Spec} B$ be the induced morphism, and let it correspond to a k-algebra homomorphism $f'^\sharp : B \to A$. If $i_A : \operatorname{Spec} A \hookrightarrow X$ and $i_B : \operatorname{Spec} B \hookrightarrow Y$ are the open embeddings, then $f \circ i_A = f|_{\operatorname{Spec} A} = i_B \circ f' : \operatorname{Spec} A \to Y$, and each of i_A, i_B, f and f' is flat of a constant relative dimension d. Recall that (1.32) we have homomorphisms of sheaves $f^\sharp : f^{-1}\mathcal{K}_Y^\times \to \mathcal{K}_X^\times$, which by statement 1.32 commute with restrictions to open subsets.

$$\begin{array}{ccc}
\mathcal{K}^\times(Y) & \stackrel{f^\sharp}{\to} & \mathcal{K}^\times(X) \\
i_B^\sharp \downarrow & & \downarrow i_A^\sharp \\
\mathcal{K}^\times(\operatorname{Spec} B) & \stackrel{f'^\sharp}{\to} & \mathcal{K}^\times(\operatorname{Spec} A)
\end{array}$$

Let $r_B \in \mathcal{K}^\times(\operatorname{Spec} B) = Q(B)$ denote the restriction $r|_{\operatorname{Spec} B}$. Therefore

$$(f^*[r])|_{\operatorname{Spec} A} = f'^*[r_B] \text{ and } (f^\sharp r)|_{\operatorname{Spec} A} = f'^\sharp(r_B).$$

Similarly by 5.7 we have $i_A^* \circ f^* = (f|_{\operatorname{Spec} A})^* = f'^* \circ i_B^* : Z_{n-1}(Y) \to Z_{n+d-1}(\operatorname{Spec} A)$. By applying Case 1 to $\operatorname{Spec} A \hookrightarrow X$, $f^\sharp r \in \mathcal{K}^\times(X)$ and to $\operatorname{Spec} B \hookrightarrow Y$, $r \in \mathcal{K}^\times(Y)$, we have

$$[f^\sharp r]|_{\operatorname{Spec} A} = [(f^\sharp r)|_{\operatorname{Spec} A}] \in Z_{n-1}(\operatorname{Spec} A) \text{ and } [r]|_{\operatorname{Spec} B} = [r_B].$$

Therefore in order to prove that the two sections $f^*[r]$, $[f^\sharp r] \in Z_{n+d-1}(X) = \Gamma(X,\, \mathcal{Z}_{X,n+d-1})$ have the same restriction to $\operatorname{Spec} A$, it is enough to show that

$$f'^*[r_B] = [f'^\sharp(r_B)] \in Z_{n+d-1}(\operatorname{Spec} A).$$

But this follows from Case 2. Now, we can take an affine open cover $(\operatorname{Spec} B_j)$ of Y, and for each j, take an affine open cover $(\operatorname{Spec} A_{i,j})$ of X. Then as i and j vary, the $\operatorname{Spec} A_{i,j}$ form an open cover of X. Hence the two sections $f^*[r]$, $[f^\sharp r] \in Z_{n+d-1}(X) = \Gamma(X,\, \mathcal{Z}_{X,n+d-1})$ have equal restrictions to each $\operatorname{Spec} A_{i,j}$, as desired. □

5.10 Diagram for Case 3.

$$\begin{array}{cccccc}
\mathcal{K}^\times(Y) & & \to & & & \mathcal{K}^\times(X) \\
& \searrow & & & \swarrow & \\
& \mathcal{K}^\times(\operatorname{Spec} B) & \to & \mathcal{K}^\times(\operatorname{Spec} A) & & \\
\downarrow & \downarrow & & \downarrow & & \downarrow \\
& Z_{n-1}(\operatorname{Spec} B) & \to & Z_{n+d-1}(\operatorname{Spec} A) & & \\
& \nearrow & & & \nwarrow & \\
Z_{n-1}(Y) & & \to & & & Z_{n+d-1}(X)
\end{array}$$

The proof shows that the composite map (northwest arrow)∘(right arrow)∘(down arrow) from $\mathcal{K}^\times(Y)$ to $Z_{n+d-1}(\operatorname{Spec} A)$ equals the composite (northwest arrow)∘(down arrow)∘(right arrow) from $\mathcal{K}^\times(Y)$ to $Z_{n+d-1}(\operatorname{Spec} A)$, by using the commutativity of various quadrilaterals in the above diagram.

5.11 Theorem: Flat pullbacks on cycles preserve rational equivalence, descend functorially to Chow groups.
Let $f : X \to Y$ be a flat morphism of schemes of constant relative dimension d. Then for each n, the pullback map $f^* : Z_n(Y) \to Z_{n+d}(X)$ maps $Rat_n(Y)$ into $Rat_{n+d}(X)$, thereby inducing a homomorphism $f^* : A_n(Y) = Z_n(Y)/Rat_n(Y) \to Z_{n+d}(X)/Rat_{n+d}(X) = A_{n+d}(X)$. The pullback f^* is functorial in f, that is, id_X^* is the identity, and if $f : X \to Y$ and $g : Y \to Z$ are flat morphisms of constant relative dimensions p and q respectively, then $(g \circ f)^* = f^* \circ g^* : A_*(Z) \to A_{*+p+q}(X)$.

Proof. (See [N], Theorem 6.) By definition, $Rat_n(Y)$ is generated by elements of the form $[r]_V$ where $V \subset Y$ is a closed subvariety of dimension $n+1$ and $r \in \mathcal{K}(V)^\times$. Hence we just have to show that $f^*[r]$ lies in $Rat_{n+d}(X)$. Let $W = f^{-1}(V) \subset X$ be the schematic inverse image of V. Hence $g = f|_W : W \to V$ is again flat and of constant relative dimension d. As V is a variety hence of pure dimension n, this implies that W is a scheme of pure dimension $n + d + 1$. Hence by Proposition 5.9,

$$g^*[r] = [g^\sharp r]$$

where $g^\sharp : \mathcal{K}^\times(V) \to \mathcal{K}^\times(W)$. By Proposition 5.8.(2) applied to the scheme W and to $g^\sharp r \in \mathcal{K}^\times(W)$, the cycle $[g^\sharp r]$ lies in the subgroup $Rat_{n+d}(W) \subset Rat_{n+d}(X)$. This proves the first assertion of the theorem.

The functoriality now follows from 5.7. This completes the proof. $\square$

6 Miscellaneous

Push-pull equals pull-push.

6.1 Proposition: Push-pull equals pull-push. *Let $f : X \to Y$ be a proper morphism and $g : Y' \to Y$ a relatively n-dimensional flat morphism of algebraic schemes. Let*

$$\begin{array}{ccc} X' & \overset{g'}{\to} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \overset{g}{\to} & Y \end{array}$$

be the resulting cartesian square. Then we have

$$g^* \circ f_* = f'_* \circ g'^* : Z_d X \to Z_{d+n} Y'.$$

Proof. It is enough to prove that if $V \subset X$ is a d-dimensional closed subvariety then $g^* \circ f_*[V] = f'_* \circ g'^*[V]$.

Case 1. Let $f : X \to Y$ be a closed embedding. Then $f' : X' \to Y'$ is again a closed embedding, and the result is clear.

Case 2. Let Y be a variety, and let $f(V) = Y$. Let $i : V \hookrightarrow X$ be the inclusion. We have two cartesian squares, one on the top of the other,

$$\begin{array}{ccc} X'' & \stackrel{g''}{\to} & V \\ f' \downarrow & & \downarrow i \\ X' & \stackrel{g'}{\to} & X \\ f' \downarrow & & \downarrow f \\ Y' & \stackrel{g}{\to} & Y \end{array}$$

which 'compose' to form the large (outer) cartesian square (rectangle). The result holds for the upper cartesian square by Case 1. If we just show that the desired conclusion holds for the large cartesian square, then it would follow that the desired conclusion holds for the lower cartesian square.

The result for the large cartesian square is clear (both sides are zero) if $\dim(V) \neq \dim f(V)$. If $\dim(V) = \dim f(V)$, we can replace $f : V \to f(V)$ by a finite extension of fields L/K, where K and L are the function fields of $f(V)$ and V respectively, and replace Y' by an artin local K-algebra A, which is the local ring $\mathcal{O}_{Y',Y'_i}$ at the generic point of any irreducible component Y'_i of Y. Then the desired conclusion follows from the algebraic facts given in statement 2.25.

Case 3. General case: First base change to $f(V) \hookrightarrow Y$, regarded as a closed subvariety of Y, and then use the above two cases in a diagram chase. $\square$

Restriction of a Cartier divisor

6.2 The group of all Cartier divisors on a scheme X is the group $\Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$. All **effective Cartier divisors** on X form its subgroup $\Gamma(X, \mathcal{S}_X/\mathcal{O}_X^\times) \subset \Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$. Equivalently, an effective Cartier divisor on a scheme X is a closed subscheme D whose ideal sheaf I is locally principal, that is, I is locally generated by a single non zero-divisor. If (U_i) is an open cover of X on which $I_{U_i} = (f_i)$ for $f_i \in \mathcal{S}_X(U_i)$, then D corresponds to element $(f_i) \in \Gamma(X, \mathcal{S}_X/\mathcal{O}_X^\times)$.

6.3 Exercises. (1) Show that the above indeed defines a bijection between the set of all closed subscheme of X whose ideal sheaf is locally principal, and the set $\Gamma(X, \mathcal{S}_X/\mathcal{O}_X^\times)$.

(2) Show that a closed subscheme $D \subset X$ is an effective Cartier divisor if and only if its ideal sheaf $I \subset \mathcal{O}_X$ is an invertible sheaf.

(3) If the closed subschemes D_1 and D_2 are effective Cartier divisors in X, then describe their sum $D_1 + D_2$ as a closed subscheme of X in terms of its ideal sheaf.

6.4 A Cartier divisor $D \in \Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$ is called a **principal Cartier divisor** if there exists an $r \in \mathcal{K}^\times(X)$ such that $D = \mathrm{div}(r) \in \Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$, the image of r under the map $\Gamma(X, \mathcal{K}_X^\times) \to \Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$. All principal Cartier divisors form the subgroup $\mathrm{im}\, \Gamma(X, \mathcal{K}_X^\times) \subset \Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$. The quotient $CaCl(X) = \Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)/\,\mathrm{im}\, \Gamma(X, \mathcal{K}_X^\times)$ is called the **group of Cartier divisor classes** on X. Under the connecting homomorphism of the long exact cohomology sequence of the short exact sequence of sheaves $0 \to \mathcal{O}_X^\times \to \mathcal{K}_X^\times \to \mathcal{K}_X^\times/\mathcal{O}_X^\times \to 0$, the group $CaCl(X)$ is naturally isomorphic to the image of the connecting homomorphism $\Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times) \to H^1(X, \mathcal{O}_X^\times) = Pic(X)$.

6.5 The Weyl divisor of a Cartier divisor. Let X be an equidimensional scheme of dimension n. Then we have a natural homomorphism $\mathcal{K}_X^\times \to \mathcal{Z}_{X,n-1}$ which associates to an invertible meromorphic function f on an open subscheme U its Weyl divisor $[f] \in Z_{n-1}(U)$. As $\mathcal{O}_X^\times \subset \mathcal{K}_X^\times$ lies in the kernel of the above homomorphism, we have an induced homomorphism of sheaves $\mathcal{K}_X^\times/\mathcal{O}_X^\times \to \mathcal{Z}_{X,n-1}$ which associates to any Cartier divisor D on an open subscheme U its Weyl divisor $[D] \in Z_{n-1}(U)$.

6.6 Exercise. If D is an effective Cartier divisor on an equidimensional scheme X of dimension n, then we have two different definitions of the Weyl divisor $[D]$. The first one treats D as a closed subscheme, all whose irreducible components D_j are of dimension $n-1$, and defines $[D]$ to be its fundamental cycle $[D] = \sum_j \ell(\mathcal{O}_{D,D_j})[D_j]$. The second definition treats D as an element of $\Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$, and takes its Weyl divisor $[D]$ as defined in statement 6.5. Show that these two definitions agree for any effective Cartier divisor D on an equidimensional scheme X.

6.7 Exercise. If the scheme X is integral, then show that the inclusion $CaCl(X) \hookrightarrow Pic(X)$ is actually a bijection. Give an example of a (non-integral, algebraic) scheme for which for which the inclusion $CaCl(X) \hookrightarrow Pic(X)$ is proper.

6.8 Let X be an equidimensional scheme of dimension n, and let Y be an irreducible component of X. We have already defined in statement 1.37 how any $r \in \mathcal{K}^\times(X)$ restricts to define elements $r_Y \in \mathcal{K}^\times(Y)$. Any nonempty open subscheme $U \subset X$ is again equidimensional of dimension n. Let $U_Y = U \cap Y$. Then either $U_Y = \emptyset$ or U_Y is an irreducible component of U. The description of the passage from r to r_Y actually shows that any $f \in \mathcal{K}^\times(U)$ defines an element $f_{U_Y} \in \mathcal{K}^\times(U_Y)$ and this is compatible with restrictions to open subschemes of X. Moreover, if $f \in \mathcal{O}^\times(U)$ then $f_{U_Y} \in \mathcal{O}^\times(U_Y)$. Hence for any Cartier divisor $D \in \Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$, defined in terms of an open cover U_j of X by functions $r_j \in \mathcal{K}_X^\times(U_j)$, by restriction of the f_j's from U_j to $U_j \cap Y$ we obtain a Cartier divisor $D_Y \in \Gamma(X_i, \mathcal{K}_Y^\times/\mathcal{O}_Y^\times)$.

6.9 Exercise: Restriction of an effective Cartier divisor to an irreducible component. Let $D \in \Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$ have restriction $D_Y \in \Gamma(X_i, \mathcal{K}_Y^\times/\mathcal{O}_Y^\times)$ as defined above. In case D is effective, let D denote the corresponding closed subvariety of X, and let $D \cap Y$ denote its schematic intersection with Y. Then show that $D_Y = D \cap Y$.

6.10 Proposition. (See [F] Lemma 1.7.2.) *Let X be an equidimensional algebraic scheme of dimension n, let the closed subscheme $D \subset X$ be an effective Cartier divisor* *on X, and let $D \cap X_i$ (schematic intersection) denote its restriction to any irreducible component X_i of X. Then*

$$[D] = \sum_i \ell(\mathcal{O}_{X,X_i})\,[D \cap X_i] \in Z_{n-1}(X)$$

Proof. Let the irreducible components D_j of D be regarded as closed subvarieties of X, each of dimension $n-1$. By definition, the fundamental cycle of D is

$$[D] = \sum_j \ell(\mathcal{O}_{D,D_j})[D_j] \in Z_{n-1}(X).$$

Note that the irreducible components of the scheme $D \cap X_i$ are all those D_j such that $D_j \subset X_i$. Therefore we have

$$[D \cap X_i] = \sum_{\{j \,|\, D_j \subset X_i\}} \ell(\mathcal{O}_{D\cap X_i, D_j})[D_j].$$

Hence

$$\sum_i \ell(\mathcal{O}_{X,X_i})\,[D \cap X_i] = \sum_{\{(i,j)\,|\,D_j \subset X_i\}} \ell(\mathcal{O}_{X,X_i})\ell(\mathcal{O}_{D\cap X_i,D_j})[D_j].$$

The result will follow if we show that the coefficient of $[D_j]$ is the same in the above sum and in $[D]$, that is, for each j we have

$$\sum_{\{i\,|\,D_j\subset X_i\}} \ell(\mathcal{O}_{X,X_i})\ell(\mathcal{O}_{D\cap X_i,D_j}) = \ell(\mathcal{O}_{D,D_j})$$

For any j, let $A = \mathcal{O}_{X,D_j}$, which is a 1-dimensional noetherian local k-algebra. All those X_i such that $D_j \subset X_i$ are in one-to-one correspondence with the minimal primes $\mathfrak{p}_i$ of A. In $\operatorname{Spec} A$, the Cartier divisor $D \cap \operatorname{Spec} A$ is defined by a principal ideal (a) where $a \in S(A)$. Then $\mathcal{O}_{X,X_i} = A_{\mathfrak{p}_i}$, $\mathcal{O}_{D\cap X_i,D_j} = A/(\mathfrak{p}_i + (a))$ and $\mathcal{O}_{D,D_j} = A/(a)$, so the above numerical equation reads

$$\sum_{\{i\,|\,D_j\subset X_i\}} \ell(A_{\mathfrak{p}_i})\,\ell\left(\frac{A}{\mathfrak{p}_i + (a)}\right) = \ell(A/(a)).$$

This holds by statement **2.28**. □

6.11 Proposition. (See [F] Remark 1.7.3.) *Let X be an equidimensional algebraic scheme of dimension n and let $D \in \Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$ be a Cartier divisor on X. For each irreducible component X_i of X, let $D_{X_i} \in \Gamma(X_i, \mathcal{K}_{X_i}^\times/\mathcal{O}_{X_i}^\times)$ denote the restriction of D to X_i. Let $[D] \in Z_{n-1}(X)$ and $[D_{X_i}] \in Z_{n-1}(X_i) \subset Z_{n-1}(X)$ be the corresponding Weil divisors. Then we have*

$$[D] = \sum_i \ell(\mathcal{O}_{X,X_i})\,[D_{X_i}] \in Z_{n-1}(X).$$

Proof The element $D \in \Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$ can be described in terms of a suitable affine open cover (U_p) of X, where $U_p = \operatorname{Spec} A_p$, by elements a_p/b_p where $a_p, b_p \in S(A_p)$. The support $|D| \subset X$ of D is a closed subset of X, and it contains only finitely many closed subvarieties $D_j \subset X$ of dimension $n-1$. For any such D_j, choose any one U_p such that $U_p \cap D_j \neq \emptyset$. Then the integer

$$\operatorname{ord}_{D_j}(D) = \operatorname{ord}_{D_j}(a_p) - \operatorname{ord}_{D_j}(b_p)$$

can be seen to be well-defined, independent of the choice of U_p. Then by definition,

$$[D] = \sum_j \mathrm{ord}_{D_j}(D)[D_j] \in Z_{n-1}(X).$$

The coefficient of $[D_j]$ in the above sum is $\mathrm{ord}_{D_j}(a_p) - \mathrm{ord}_{D_j}(b_p)$.
On the other hand

$$[D_{X_i}] = \sum_{\{j \,|\, D_j \subset X_i\}} \mathrm{ord}_{D_j}(D_{X_i})[D_j].$$

Hence the coefficient of D_j in the right hand side of the equality asserted by the Proposition is

$$\sum_{\{i \,|\, D_j \subset X_i\}} \ell(\mathcal{O}_{X,X_i})\,\mathrm{ord}_{D_j}(D_{X_i}).$$

For any irreducible component X_i of X, the restriction D_{X_i} is described in terms of its open cover $(U_p \cap X_i)$ (which is again an affine open cover, as $U_p \cap X_i$ is closed in the affine U_p) by $((a_p|_{X_i})/(b_p|_{X_i}))$. Hence the coefficient of D_j becomes $\sum_{\{i \,|\, D_j \subset X_i\}} \mathrm{ord}_{D_j}(a_p|_{X_i}) - \mathrm{ord}_{D_j}(b_p|_{X_i})$. In order to prove the Proposition, we just have to prove that the coefficients of $[D_j]$ on the two sides are equal, that is, we must show that

$$\mathrm{ord}_{D_j}(a_p) - \mathrm{ord}_{D_j}(b_p) = \sum_{\{i \,|\, D_j \subset X_i\}} \mathrm{ord}_{D_j}(a_p|_{X_i}) - \mathrm{ord}_{D_j}(b_p|_{X_i}).$$

As in the course of the proof of the Proposition 6.10, we can separately show the corresponding equalities for a_p and b_p, by appeal to the statement 2.28, applied to the 1-dimensional noetherian local ring $\mathcal{O}_{X,D_j}$, whose minimal primes correspond to the irreducible components X_i which contain D_j. Subtracting the equality for b_p from that for a_p gives us the result. $\square$

Remark. The Proposition 6.11 (which is [F] Example 1.7.3) also gives a proof of the Proposition 6.10, (which is [F] Lemma 1.7.2) as a special case. In [F], before stating Lemma 1.7.2, the restriction of an effective Cartier divisor on X to an irreducible component X_i is defined by treating an effective Cartier divisor as a closed subscheme. The notion of the restriction of a (not necessarily effective) Cartier divisor on X to an irreducible component X_i is not defined before stating Example 1.7.3, for which no proof is given in [F] at that stage of the book. A more general discussion on Cartier divisors and their restrictions to subvarieties appears later in Chapter 2 the book.

Open-closed right-exact sequence.

6.12 ([F] Proposition 1.8.) *Let $i : Y \hookrightarrow X$ be a closed subscheme of a scheme X, and let $j : U = X - Y \hookrightarrow X$ be its complementary open subscheme of X. Then the following sequence is exact for all n.*

$$A_n(Y) \xrightarrow{i_*} A_n(X) \xrightarrow{j^*} A_n(U) \to 0$$

Proof (See [F]). By Exercise 3.6, the disjoint union $X_{(n)} = Y_{(n)} \coprod U_{(n)}$ as sets gives rise to a direct sum decomposition $Z_n(X) = Z_n(Y) \oplus Z_n(U)$ where the inclusion $Z_n(Y) \hookrightarrow Z_n(X)$ is the homomorphism i_* corresponding to the proper morphism i, and the projection

$Z_n(X) \to Z_n(U)$ is the homomorphism j^* corresponding to the flat morphism j. In particular, the cycle-level sequence

$$0 \to Z_n(Y) \xrightarrow{i_*} Z_n(X) \xrightarrow{j^*} Z_n(U) \to 0$$

is exact.

From the above, it follows that in the sequence $A_n(Y) \xrightarrow{i_*} A_n(X) \xrightarrow{j^*} A_n(U) \to 0$, the map j^* is surjective, and $j^* \circ i_* = 0$. It only remains to show that if $\alpha \in Z_n(X)$ is such that $j^*\alpha \in Rat_n(U)$, then $\alpha \in Rat_n(X) + i_*Z_n(Y)$.

If $j^*\alpha \in Rat_n(U)$, then we have $j^*\alpha = \sum_c [r_c]$ where each $r_c \in R(W_c)^\times$ is a nonzero rational function on a closed subvariety $W_c \subset U$. Let $\overline{W_c} \subset X$ denote the closure of W_c, regarded as a closed subvariety of X. As the inclusion $W_c \hookrightarrow \overline{W_c}$ is birational, r_c is the restriction of a nonzero rational function $r'_c \in R(\overline{W_c})^\times$. Then $j^*[r'_c] = [r_c]$. Hence for $\beta = \alpha - \sum_c [r'_c] \in Z_n(X)$, we have $j^*(\beta) = 0 \in Z_n(U)$. By exactness of $0 \to Z_n(Y) \xrightarrow{i_*} Z_n(X) \xrightarrow{j^*} Z_n(U)$ it follows that $\beta = i_*\gamma$ for some (actually, a unique) $\gamma \in Z_n(Y)$. Hence $\alpha = \sum_c [r'_c] + i_*\gamma$ with $\sum_c [r'_c] \in Rat_n(X)$ and $\gamma \in Z_n(Y)$. This proves the result. $\square$

6.13 ([F] Example 1.8.1.) Consider a cartesian square

$$\begin{array}{ccc} Y' & \xrightarrow{j} & X' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{i} & X \end{array}$$

where i is a closed embedding (hence so is j) and p is proper (hence so is q). Suppose that p induces an isomorphism $X' - Y' \to X - Y$. Then the following sequence is exact

$$A_k(Y') \xrightarrow{a} A_k(Y) \oplus A_k(X') \xrightarrow{b} A_k(X) \to 0$$

where the homomorphisms a and b are defined by

$$a(\alpha) = (q_*\alpha,\ -j_*\alpha) \text{ and } b(\beta, \gamma) = i_*\beta + p_*\gamma.$$

Proof The hint given in [F] is to use the definitions of cycles and rational equivalence to show that under $q_* : A_*(Y') \to A_*(Y)$, the subgroup $\ker(j_*) \subset A_*(Y')$ surjects onto $\ker(i_*) \subset A_*(Y)$. To prove this surjection, I actually need to use something more, namely Proposition 4.6. Am I missing something?

In order to prove that $\ker(j_*)$ surjects onto $\ker(i_*)$, consider the following commutative diagram, whose rows are exact by the Proposition 6.12.

$$\begin{array}{ccccccccc} 0 \to & \ker(j_*) & \to & A_n(Y') & \xrightarrow{j_*} & A_n(X') & \to & A_n(X'-Y') & \to 0 \\ & \downarrow & & \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle p_*} & & \| & \\ 0 \to & \ker(i_*) & \to & A_n(Y) & \xrightarrow{i_*} & A_n(X) & \to & A_n(X-Y) & \to 0 \end{array}$$

Suppose that $\alpha \in Z_n(Y)$ defines an element $(\alpha) \in A_n(Y)$ of $\ker(i_*)$. Then in $Z_n(X)$ we must have $\alpha = \sum [r_i]_{W_i}$ where W_i are $n+1$-dimensional closed subvarieties of X and

$r_i \in R(W_i)^\times$. If $W_i \subset Y$ for some i, then we can replace $\alpha \in Z_n(Y)$ by $\alpha - [r_i]_{W_i} \in Z_n(Y)$, which defines the same element $(\alpha) \in A_n(X)$. By repeating this device, we can assume that $W_i \cap (X - Y) \neq \emptyset$ for each i. As $p : X' - Y' \to X - Y$ is an isomorphism, for each i there exists a unique $n+1$-dimensional closed subvariety W_i' of X' such that $W_i' \cap (X' - Y')$ maps isomorphically under $p_{X-Y} : X' - Y' \to X - Y$ to $W_i \cap (X - Y)$, and W_i' maps birationally onto W_i under $p : X' \to X$. Hence the induced map $R(W_i) \to R(W_i')$ is an isomorphism. Let $r_i \mapsto r_i' \in R(W_i')^\times$. Now take $\alpha' = \sum [r_i'] \in Z_n(X')$. Then as $\alpha'|_{X'-Y'} = 0 \in Z_n(X' - Y')$, by exactness of $Z_n(Y') \to Z_n(X') \to Z_n(X' - Y') \to 0$, we have $\alpha' \in Z_n(Y')$. Note that $(\alpha') \in \ker(j_*)$ as $\alpha'|_{X'-Y'} = 0$.

We now show that $(\alpha') \mapsto (\alpha)$ under $q_* : A_n(Y') \to A_n(Y)$. For this it is enough to show that $q_*[r_i'] = [r_i]$. But this follows from applying Proposition 4.6 to the surjection $W_i' \to W_i$.

Hence $\ker(j_*)$ surjects onto $\ker(i_*)$, as stated in the hint. The rest is now just a diagram chase using the commutative diagram above, which is left to the reader. □

Noetherian induction

6.14 Let $(S, \leq)$ be a poset (partially ordered set) which is noetherian, that is, any nonempty subset of S has a minimal element. If T is a subset of S and $x \in S$, then we will call the following statement $C(T, x)$ as the **noetherian inductive hypothesis** on x for T.

$C(T, x)$: For all $y \in S$, if $y < x$ then $y \in T$. More symbolically, $C(T, x)$ is the statement $(\forall y \in S)(y < x \Rightarrow y \in T)$.

Proposition: Noetherian induction. *Let $(S, \leq)$ be a noetherian poset and let T be a subset of S. Suppose that for all $x \in S$, we have $C(T, x) \Rightarrow x \in T$. Then $T = S$.*

Proof: If $T \neq S$, then $S - T$ has a minimal element x. Then $C(T, x)$ is satisfied. Therefore $x \in T$, which is a contradiction. □

Note: Hence the main work to be done in a proof by noetherian induction is to prove the implication "$C(T, x) \Rightarrow x \in T$" for every $x \in S$. We will call this as the **(noetherian) inductive step** for x.

6.15 Exercise. Suppose we have a property P that is meaningful for all noetherian schemes (or for all algebraic schemes over a chosen ground field k). Then to show that all noetherian schemes (resp. all algebraic k-schemes) have this property, it is enough to show that if X is a noetherian scheme (resp. an algebraic k-scheme) such that every proper closed subscheme $Y \subset X$ (where 'proper' means $Y \neq X$) has the property P, then X has the property P. (Hint: Let S be the poset of all closed subschemes of X and let $T \subset S$ consist of all Y that satisfy P. Now apply the above proposition.)

The above is used in Fulton [F] Section 1.9 for the property P of an algebraic k-scheme X which says that if $\pi : E \to X$ is a Zariski locally trivial $\mathbb{A}^n_k$-fibration then the pullback homomorphism $\pi^* : A_*(X) \to A_{*+n}(E)$ is surjective.

Chow group of the total space of an affine fibration.

6.16 An **affine fibration of rank** n, where $n \geq 0$, is a morphism $\pi : E \to X$ of schemes such that for each $x \in X$ there exists an open subscheme $U \subset X$ with $x \in U$, and an isomorphism $\phi : \pi^{-1}(U) \to U \times \mathbb{A}^n_k$ over U. Note that if $\pi : E \to X$ is an affine fibration of rank n, and if $X \neq \emptyset$, then n is unique. Any affine fibration is a flat (in fact, smooth) morphism of constant relative dimension n. Hence we have an induced pullback homomorphism $\pi^* : A_*(X) \to A_{*+n}(E)$ of graded degree n on the Chow groups. Note that if $\pi : E \to X$ is an affine fibration of rank n and if $f : Y \to X$ is any morphism of schemes, then the base-change $\pi_Y : E_Y \to Y$ of π under f is again an affine fibration of rank n.

6.17 Proposition. *If $\pi : E \to X$ is an affine fibration of rank n, then the pullback homomorphism $\pi^* : A_*(X) \to A_{*+n}(E)$ is surjective.*

6.18 For future reference. If $\pi : E \to X$ is a vector bundle or rank n, then $\pi^* : A_*(X) \to A_{*+n}(E)$ is in fact an isomorphism (see [F] Theorem 3.3). In particular, $\pi^* : A_*(X) \to A_{*+n}(\mathbb{A}^n_X)$ is an isomorphism. We will not prove it here. Instead, we will prove only the weaker Lemma 6.19 which says that π^* is surjective.

Question. Is π^* an isomorphism for all affine fibrations? Proof or counterexample?

6.19 Lemma: Surjectivity of pullback for a trivial bundle. Let X be a scheme and $\pi : \mathbb{A}^n_X = X \times \mathbb{A}^n_k \to X$ be the first projection, where $n \geq 0$. Then the pullback homomorphism $\pi^* : A_*(X) \to A_{*+n}(\mathbb{A}^n_X)$ is surjective.

Proof By factorizing the projection π as a composite $\pi = p_0 \circ p_1 \circ \ldots \circ p_n$ of $n+1$ projections $p_i : X_{i+1} = \mathbb{A}^1_{X_i} \to X_i$, where $X_0 = X$ and X_n is isomorphic to $\mathbb{A}^n_X$ over X, it follows that to prove the lemma in general, it is enough to prove it in the case $n = 1$.

We prove this by noetherian induction (see Exercise 6.15). The inductive hypothesis on X is that $\pi_Y^* : A_*(Y) \to A_{*+1}(\mathbb{A}^1_Y)$ is surjective for all proper closed subschemes $Y \subset X$. Under this hypothesis on X, we have to show that $\pi^* : A_*(X) \to A_{*+n}(\mathbb{A}^1_X)$ is surjective.

We will use the notation $\pi : E \to X$ for $\pi : \mathbb{A}^1 \to X$. To prove the surjectivity of $A_m(X) \to A_{m+1}(E)$, we just have to prove that if $V \subset E$ is a closed subvariety of dimension $m+1$, then $[V]$ lies in the image of $\pi^* : A_m(X) \to A_{m+1}(E)$. Let $Y \subset X$ be the closure of the set-theoretic image $\pi(V)$ in X with reduced induced subscheme structure. As V is irreducible it can be seen that Y is irreducible, and hence it is a subvariety of X. As $V \to Y$ is dominant, it follows that $\dim(Y) \leq \dim(V) = m+1$.

Now there are two cases possible: (1) $Y \neq X$ and (2) $Y = X$.

If $Y \neq X$, then by inductive hypothesis the projection $\pi_Y : E_Y \to Y$ (where $E_Y = \mathbb{A}^1_Y$) induces a surjection $\pi_Y^* : A_*(Y) \to A_{*+1}(E_Y)$. Hence there exists some $\alpha \in A_*(Y)$ such that $\pi_Y^*(\alpha) = [V]_{E_Y}$.

The following square is cartesian

$$\begin{array}{ccc} E_Y & \stackrel{i'}{\to} & E \\ {\scriptstyle \pi_Y}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \stackrel{i}{\to} & X \end{array}$$

where $i' : E_Y \to E$ and $i : Y \to X$ are the inclusions, which are proper, and π_Y and π are the projections, which are flat. Hence by Proposition 6.1 ('push-pull equals pull-push'), we have

$$\pi^* i_*(\alpha) = i'_* \pi_Y^*(\alpha).$$

As $\pi_Y^*(\alpha) = [V]_{E_Y}$ and $i'_*([V]_{E_Y}) = [V]_E$, the above reads $\pi^* i_*(\alpha) = [V]_E$, showing that $[V]_E$ is in the image of π^* in case $Y \neq X$.

So now consider the remaining case, where $Y = X$. In particular, this means that X is a variety. This means E is also a variety.

Now there are two possibilities: either $\dim(X) = m$, or $\dim(X) = m+1$. If $\dim(X) = m$, then as V is a closed subvariety of E of dimension $m+1$ and as $\dim(E) = \dim(X) + 1 = m+1$, we must have $V = E$. Hence $[V] = [E] = \pi^*[X]$.

It now only remains to consider the case where $\dim(X) = m+1$. Then the generic point ξ of V is a closed point of the fiber E_η of $E \to X$ over the generic point η of the variety X. As E_η is isomorphic to the affine line $\mathbb{A}^1_K = \operatorname{Spec} K[t]$ over the field $K = R(X)$, the closed point $\eta \in \operatorname{Spec} K[t]$ is defined by a maximal ideal, which is a principal ideal $(f) \subset K[t]$, where f is irreducible of degree ≥ 1. Let $U = \operatorname{Spec} A$ be a nonempty affine open subscheme of X. Then A is a domain with quotient field K, so we can choose $f \in A[t]$ (which may no more be monic). Then the closed subscheme $E_U/(f)$, which is an effective principal Cartier divisor, restricts to the closed point ξ over η, which is also the restriction of V' over η, where $V' = V \cap E_U$ (note that V' is a closed subvariety of E_U of dimension $m+1$). Hence $V' \subset E_U/(f)$, showing that V' is an irreducible component of $E_U/(f)$. If $E_U/(f)$ is irreducible, then this means V' is the only irreducible component of $E_U/(f)$. Otherwise $E_U/(f)$ may have some more irreducible components, say $V_1, \ldots, V_r$. Then $[f] = [V'] + \sum n_i [V_i]$ where each $n_i > 0$ (see 2.16 for the reason why n_i is > 0). The coefficient of $[V']$ is 1, because $E_U/(f)$ restricts to ξ over η. Moreover, V_i does not intersect E_η, as $E_U/(f)$ restricts to ξ over η. Hence the image $\pi_U(V_i)$ does not contain η. As the image is constructible, it follows that $W_i = \overline{\pi_U(V_i)}$ is a proper subvariety of U, and so it has dimension $< m+1$. As the fiber dimension of $E_U \to U$ is 1, it follows that $\dim(W_i) = m$. As $V_i \subset \pi_U^{-1}(W_i)$, and both are closed subvarieties of dimension $m+1$, we must have $V_i = \pi_U^{-1}(W_i)$. This shows that $[V_i] = \pi_U^*[W_i] \in Z_{m+1}(E_U)$. Hence $[V'] = [f] - \sum \pi_U^*[W_i]$, and so $[V']$ lies in the image of $A_m(U)$.

Now let $Z = X - U$, which we regard as a closed subscheme of X with reduced induced subscheme structure. As $Z \neq X$, by noetherian inductive hypothesis, the map $\pi_Z^* : A_*(Z) \to A_{*+1}(E_Z)$ is surjective, where $E_Z = \pi^{-1}(Z) \subset E$.

Consider the following commutative diagram.

$$\begin{array}{ccccccc} A_{m+n}(E_Z) & \to & A_{m+n}(E) & \to & A_{m+n}(E_U) & \to 0 \\ {\scriptstyle \pi_Z^*}\uparrow & & {\scriptstyle \pi^*}\uparrow & & {\scriptstyle \pi_U^*}\uparrow & \\ A_m(Z) & \to & A_m(X) & \to & A_m(U) & \to 0 \end{array}$$

The rows are induced by the proper embeddings $Z \hookrightarrow X$ and $E_Z \hookrightarrow E$ and complementary open embeddings $U \hookrightarrow X$ and $E_U \hookrightarrow E$. Both the rows are exact by Proposition 6.12. The element $[V] \in A_{m+n}(E)$ maps to $[V'] \in A_{m+n}(E_U)$ which lies in the image of π_U^*, while π_Z^* is surjective. Hence it follows from a diagram chase that $[V]$ lies in the image of $A_m(X)$. $\square$

6.20 Exercise: Chow groups of an affine space. Use Lemma 6.19 to prove that for any field k, the affine space $\mathbb{A}_k^n$ (where $n \geq 1$) has $A_n(\mathbb{A}_k^n) = \mathbb{Z}$, generated by $[\mathbb{A}_k^n]$, while $A_m(\mathbb{A}_k^n) = 0$ for all $m \neq n$. In particular, $\pi^* : A_*(\operatorname{Spec} k) \to A_{*+n}(\mathbb{A}_k^n)$ is an isomorphism.

Proof of Proposition 6.17. The result is obvious for $X = \emptyset$. Now let $X \neq \emptyset$. We proceed by noetherian induction. By local triviality, X has a nonempty open subscheme U such that $E_U \to U$ is isomorphic to the trivial fibration $\mathbb{A}_U^n \to U$. Hence by Lemma 6.19, the map $\pi_U^* : A_*(U) \to A_{*+n}$ is surjective. Let $Z = X - U$ with the reduced induced closed subscheme structure. Then by noetherian inductive hypothesis, the map $\pi_Z^* : A_*(Z) \to A_{*+n}(E_Z)$ is surjective. We have the following commutative diagram whose rows are exact by Proposition 6.12.

$$\begin{array}{ccccccc}
A_{m+n}(E_Z) & \to & A_{m+n}(E) & \to & A_{m+n}(E_U) & \to 0 \\
\pi_Z^* \uparrow & & \pi^* \uparrow & & \pi_U^* \uparrow & \\
A_m(Z) & \to & A_m(X) & \to & A_m(U) & \to 0
\end{array}$$

As π_U^* and π_Z^* are surjective, a diagram-chase shows that $\pi^* : A_m(X) \to A_{m+n}(E)$ is surjective. $\square$

Exterior Products

The following is an account of the material in [F] Section 1.10 'Exterior Products', together with some extra details.

Let X, Y be schemes, which for us mean algebraic k-schemes. Then $X \times Y$, which denotes $X \times_{\operatorname{Spec} k} Y$ for simplicity, is again an algebraic k-scheme. We define a homomorphism of abelian groups

$$Z_m(X) \otimes Z_n(Y) \to Z_{m+n}(X \times Y)$$

as follows, where the tensor product on the left is that in the category of $\mathbb{Z}$-modules. For this, we first define a bi-additive map $Z_m(X) \times Z_n(Y) \to Z_{m+n}(X \times Y)$ as follows. Given closed subvarieties $V \subset X$ and $W \subset Y$, we send $([V], [W]) \in Z_m(X) \times Z_n(Y)$ to $[V \times W] \in Z_{m+n}(X \times Y)$, which is the fundamental cycle of the closed subscheme $V \times W \subset X \times Y$. Note that if k is algebraically closed, then $V \times W$ is a closed subvariety of $X \times Y$, but if k is not algebraically closed, the scheme $V \times W$ may fail to be an integral scheme. The above prescription extended by bi-additivity defines a bi-additive map $Z_m(X) \times Z_n(Y) \to Z_{m+n}(X \times Y)$, which induces the homomorphism $\times : Z_m(X) \otimes Z_n(Y) \to Z_{m+n}(X \times Y)$ which is called the **exterior product**.

6.21 Proposition. *For any schemes X and Y, the exterior product $\times : Z_m(X) \otimes Z_n(Y) \to Z_{m+n}(X \times Y)$ satisfies the following properties.*

(a) If $\alpha \in Z_m(X)$ and $\beta \in Z_n(Y)$ are such that $\alpha \sim 0$ or $\beta \sim 0$, then $\alpha \times \beta \sim 0$ in $Z_{m+n}(X \times Y)$.

(b) Let $f : X' \to X$ and $g : Y' \to Y$ be morphisms of schemes, and let $f \times g : X' \times Y' \to X \times Y$ be their product morphism. Then we have

(i) If both f and g are proper morphisms, then so is $f \times g$, and

$$(f \times g)_*(\alpha \times \beta) = f_*(\alpha) \times g_*(\beta)$$

for all $\alpha \in Z_m(X')$ and $\beta \in Z_n(Y')$.

(ii) If f and g are flat morphisms of constant relative dimensions d and e, then then $f \times g$ is a flat morphism of constant relative dimension $d + e$, and

$$(f \times g)^*(\alpha \times \beta) = f^*(\alpha) \times g^*(\beta)$$

for all $\alpha \in Z_m(X)$ and $\beta \in Z_n(Y)$.

Consequently, we have an induced homomorphism $\times : A_m(X) \otimes A_n(Y) \to A_{m+n}(X \times Y)$, which satisfies (b)(i) and (b)(ii) for any $\alpha \in A_m(X)$ and $\beta \in A_n(Y)$.

Proof We first prove (b) for $\alpha \in Z_m(X')$ and $\beta \in Z_n(Y')$. As $f \times g = (f \times \mathrm{id}_Y) \circ (\mathrm{id}_{X'} \times g) : X' \times Y' \to X' \times Y \to X \times Y$, it is enough to prove (b) in the case when one of f and g is identity. By symmetry, we can assume that $Y' = Y$ and $g = \mathrm{id}_Y$.

We now prove (b) (i), with $f : X' \to X$ proper, $Y' = Y$ and $g = \mathrm{id}_Y$. By additivity, it is enough to consider the case where $\beta = [W]$ for a closed subvariety $W \subset Y' = Y$ of dimension n. Then the structure morphism $W \to \operatorname{Spec} k$ is flat of relative dimension n, and hence so are all its base changes. Moreover, is enough to take $Y = W$. Consider the Cartesian square

$$\begin{array}{ccc} X' \times W & \stackrel{q}{\to} & X' \\ {\scriptstyle f \times \mathrm{id}_W} \downarrow & & \downarrow {\scriptstyle f} \\ X \times W & \stackrel{p}{\to} & X \end{array}$$

in which the horizontal arrows are the respective projections, which are flat of relative dimension n, while the vertical arrows are proper. Hence by the Proposition 6.1 ('push-pull equals pull-push') we have

$$(f \times \mathrm{id}_W)_* \circ q^* = p^* \circ f_* : Z_m(X) \to Z_{m+n}(X \times W).$$

On applying to $\alpha \in Z_m(X')$, the above gives

$$(f \times \mathrm{id}_W)_*(\alpha \times [W]) = f_*(\alpha) \times [W]$$

as desired.

Next we prove (b)(ii). If $\alpha = [V] \in Z_m(X)$ and $\beta = [W] \in Z_n(Y)$ correspond to closed subvarieties $V \subset X$ and $W \subset Y$, then $(f \times g)^*([V] \times [W]) = (f \times g)^{-1}[V \times W] =$

$[(f \times g)^{-1}(V \times W)]$ where the last equality is by Proposition 5.6 on the pull-back for fundamental cycles. Now, $(f \times g)^{-1}(V \times W) = f^{-1}V \times g^{-1}W$ as schemes. Hence we get $(f \times g)^*([V] \times [W]) = [f^{-1}V \times g^{-1}W] = [f^{-1}V] \times [g^{-1}W]$ where the final equality is by Exercise 6.23 below.

Now we prove (a). Let $\alpha \sim 0$. To show that $\alpha \times \beta \sim 0$, it is enough by additivity to assume that $\beta = [W]$ for a closed subvariety $W \subset Y$, and to moreover assume that $W = Y$. Note that as W is a variety, the projection $p : X \times W \to X$ is flat of relative dimension $n = \dim(W)$. Then $\alpha \times \beta = \alpha \times [W] = p^*(\alpha)$ where $p : X \times W \to X$ is the projection. As a pull-back by a flat morphism of constant relative dimension preserves rational equivalence by Theorem 5.11, it follows from $\alpha \sim 0$ that $p^*(\alpha) \sim 0$, as desired. $\square$

6.22 Exercise. Let A and B be finite type algebras over a field k, and let $\mathfrak{p} \subset A$ and $\mathfrak{q} \subset B$ be minimal prime ideals. Let a and b denote the multiplicities of $\mathfrak{p}$ in A and $\mathfrak{q}$ in B, which by definition are the lengths of the Artin local rings $A_{\mathfrak{p}}$ and $B_{\mathfrak{q}}$. Let $\mathfrak{r} \subset C = (A/\mathfrak{p}) \otimes_k (B/\mathfrak{q})$ be a minimal prime ideal, and let c denote its multiplicity in C. Let $\mathfrak{r}' \subset A \otimes_k B$ denote the contraction of $\mathfrak{r}$ under the quotient homomorphism $A \otimes_k B \to C$ and let c' denote the multiplicity of $\mathfrak{r}'$ in $A \otimes_k B$. Then show that $abc = c'$.

6.23 Exercise. If V and W are equidimensional schemes of dimensions m and n, then show that

$$[V] \times [W] = [V \times W] \in Z_{m+n}(V \times W).$$

Proof. Let V_i be the irreducible components of V and W_j those of W. These are closed subvarieties of V and W of dimensions m and n. Then $[V] = \sum_i \ell(\mathcal{O}_{V,V_i})\,[V_i]$ and $[W] = \sum_j \ell(\mathcal{O}_{W,W_j})\,[W_j]$. Hence $[V] \times [W] = \sum_{i,j} \ell(\mathcal{O}_{V,V_i})\ell(\mathcal{O}_{W,W_j})\,[V_i \times W_j]$. Let $C_{i,j,p}$ be the irreducible components of $V_i \times W_j$. These are closed subvarieties of $V \times W$ of dimension $m + n$. Then $[V_i \times W_j] = \sum_p \ell(\mathcal{O}_{V_i \times W_j, C_{i,j,p}})\,[C_{i,j,p}]$. Putting these together, we have

$$[V] \times [W] = \sum_{i,j,p} \ell(\mathcal{O}_{V,V_i})\ell(\mathcal{O}_{W,W_j})\,\ell(\mathcal{O}_{V_i \times W_j, C_{i,j,p}})\,[C_{i,j,p}] \in Z_{m+n}(X \times Y).$$

On the other hand, $C_{i,j,p}$ are also the irreducible components of $V \times W$. Hence

$$[V \times W] = \sum_{i,j,p} \ell(\mathcal{O}_{V \times W, C_{i,j,p}})\;[C_{i,j,p}] \in Z_{m+n}(X \times Y).$$

Thus, the desired conclusion is true if and only if the numerical coefficients of each $[C_{i,j,p}]$ on the two sides are equal to each other, that is, if and only if we have

$$\ell(\mathcal{O}_{V,V_i})\ell(\mathcal{O}_{W,W_j})\ell(\mathcal{O}_{V_i \times W_j, C_{i,j,p}}) = \ell(\mathcal{O}_{V \times W, C_{i,j,p}})$$

for all i, j, p. But this is the assertion of the Exercise 6.22.

6.24 Exercise: Associativity of exterior product. Using the above exercise, show that if X, Y, Z are schemes, and if $\alpha \in Z_\ell(X)$, $\beta \in Z_m(X)$ and $\gamma \in Z_n(Z)$, then

$$(\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma) \in Z_{\ell+m+n}(X \times Y \times Z).$$

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