Lectures on exponential sums

by Stephan Baier, JNU

1. Lecture 1 - Introduction to exponential sums, Dirichlet divisor problem

The main reference for these lecture notes is [4].

1.1. **Exponential sums.** Throughout the sequel, we reserve the notation I for an interval $(a,b]$, where a and b are integers, unless stated otherwise.

Exponential sums are sums of the form

$$\sum_{n\in I} e(f(n)),$$

where we write

$$e(z) := e^{2\pi iz},$$

and f is a real-valued function, defined on the interval $I=(a,b]$. The function f is referred to as the amplitude function. Note that the summands $e(f(n))$ lie on the unit circle. In applications, f will have some "nice" properties such as to be differentiable, possibly several times, and to satisfy certain bounds on its derivatives. Under suitable conditions, we can expect cancellations in the sum $\sum_{n\in I} e(f(n))$. The object of the theory of exponential sums is to detect such cancellations, i.e. to bound the said sum non-trivially. By the triangle inequality, the trivial bound is

$$\sum_{n\in I} e(f(n)) \ll |I|,$$

where $|I| = b-a$ is the length of the interval $I=(a,b]$.

Exponential sums play a key role in analytic number theory. Often, error terms can be written in terms of exponential sums. Important examples for applications of exponential sums are the Dirichlet divisor problem (estimates for the average order of the divisor function), the Gauss circle problem (counting lattice points enclosed by a circle) and the growth of the Riemann zeta function in the critical strip. We will sketch significant parts of the theory and focus on the Dirichlet divisor problem as an application.

1.2. **The Dirichlet divisor problem.** By $d(n)$, we denote the number of the divisors of the natural number n. We call the function $d(n)$ "divisor function". This function is very irregularly distributed. For example, at the primes p, we have $d(p)=2$. On the other hand, if n

has a lot of prime divisors, then $d(n)$ can be fairly large. Given the prime factorization

$$n = p_1^{\alpha_1} \cdots p_r^{\alpha_r},$$

we have the well-known formula

$$d(n) = (\alpha_1 + 1) \cdots (\alpha_r + 1). \tag{1.1}$$

Using this, we shall first prove the following bound for $d(n)$.

Lemma 1. *For every given $\varepsilon > 0$, we have*

$$d(n) \ll n^{\varepsilon}$$

as $n \to \infty$.

Proof. From (1.1), we deduce that

$$\frac{d(n)}{n^{\varepsilon}} = \prod_{k=1}^{r} \frac{\alpha_k + 1}{p_k^{\varepsilon \alpha_k}}.$$

If $p_k > \exp(1/\varepsilon)$, then

$$p_k^{\varepsilon \alpha_k} \geqslant \exp(\alpha_k) \geqslant 1 + \alpha_k.$$

It follows that

$$\prod_{k=1}^{r} \frac{\alpha_k + 1}{p_k^{\varepsilon \alpha_k}} \leqslant \prod_{\substack{k=1 \\ p_k \leqslant \exp(1/\varepsilon)}}^{r} \frac{\alpha_k + 1}{p_k^{\varepsilon \alpha_k}}.$$

On the other hand, there exists a constant C depending on ε such that

$$\frac{\alpha + 1}{p_k^{\varepsilon \alpha}} \leqslant C$$

for all $\alpha \geqslant 1$ and primes $p_k \geqslant 1$, and the number of primes $p \leqslant \exp(1/\varepsilon)$ is bounded. Hence $d(n)/n^{\varepsilon}$ is bounded, which implies the result. □

Much more can be said about the *average* of the divisor function $d(n)$, i.e. about its summatory function

$$D(x) = \sum_{n \leqslant x} d(n).$$

We will see soon that

$$D(x) = x \log x + (2\gamma - 1)x + \Delta(x)$$

with

$$\Delta(x) = O\left(x^{1/2}\right),$$

where γ is the Euler constant. Improving the above bound for $\Delta(x)$ is referred to as the Dirichlet divisor problem. Using basic results from the theory of exponential sums, we will first show that the exponent $1/2$ above can be replaced by $1/3 + \varepsilon$. Then, using more elaborate tools from this theory, we shall slightly lower this exponent further. It is conjectured that

$$\Delta(x) = O\left(x^{1/4+\varepsilon}\right).$$

By a result of Littlewood, the exponent 1/4 above is the limit, i.e. the above bound becomes false if 1/4 is replaced by any smaller value.

1.3. **A basic estimate for $D(x)$.** We first want to prove a much weaker asymptotic estimate, namely the following.

Lemma 2. *We have*

$$D(x) = x \log x + O(x). \tag{1.2}$$

Proof. Our starting point is to write

$$D(x) = \sum_{\substack{s,t \\ st \leqslant x}} 1 = \sum_{1 \leqslant s \leqslant x} \sum_{1 \leqslant t \leqslant x/s} 1 = \sum_{1 \leqslant s \leqslant x} \left[\frac{x}{s} \right], \tag{1.3}$$

where $[z]$ denotes the integral part of z, i.e. the largest integer less or equal z. We have

$$z - 1 < [z] \leqslant z$$

and hence

$$\sum_{1 \leqslant s \leqslant x} \left(\frac{x}{s} - 1 \right) < \sum_{1 \leqslant s \leqslant x} \left[\frac{x}{s} \right] \leqslant \sum_{1 \leqslant s \leqslant x} \frac{x}{s}.$$

Further, the integral test shows that

$$\log x = \int_1^x \frac{1}{s}\, ds \leqslant \sum_{1 \leqslant s \leqslant x} \frac{1}{s} \leqslant \int_1^x \frac{1}{s}\, ds + 1 = \log x + 1. \tag{1.4}$$

Combining everything, we obtain the desired result. □

1.4. **A refined estimate for $D(x)$.** From (1.3), we see that $D(x)$ equals the number of lattice points below the hyperbola $h(s) = x/s$ for $s > 0$. To refine the estimate (1.2), we use the symmetry of this region together with the following refinement of (1.4).

Lemma 3. *For $y \in \mathbb{R}$, define*

$$\psi(y) := y - [y] - 1/2.$$

Then we have

$$\sum_{s \leqslant y} \frac{1}{s} = \log y + \gamma - \frac{\psi(y)}{y} + O\left(\frac{1}{y^2} \right).$$

Proof. See the proof of Lemma 4.4 in [4]. Deferred to the tutorials. □

We are now able to prove the estimate announced in subsection 1.2.

Lemma 4. *We have*

$$D(x) = x \log x + (2\gamma - 1)x + \Delta(x)$$

with

$$\Delta(x) = O\left(x^{1/2} \right). \tag{1.5}$$

Proof. By symmetry, we have

$$D(x) = \sum_{\substack{s,t \\ st\leqslant x}} 1 = \sum_{1\leqslant s\leqslant\sqrt{x}} \sum_{1\leqslant t\leqslant x/s} 1 + \sum_{1\leqslant t\leqslant\sqrt{x}} \sum_{1\leqslant s\leqslant x/t} 1 - \sum_{1\leqslant s\leqslant\sqrt{x}} \sum_{1\leqslant t\leqslant\sqrt{x}} 1$$
$$= 2 \sum_{1\leqslant s\leqslant\sqrt{x}} \left[\frac{x}{s}\right] - [\sqrt{x}]^2.$$

Writing $[y] = y - \psi(y) - 1/2$, it follows that

$$D(x) = 2 \sum_{1\leqslant s\leqslant\sqrt{x}} \left(\frac{x}{s} - \psi\left(\frac{x}{s}\right) - \frac{1}{2}\right) - \left(\sqrt{x} - \psi(\sqrt{x}) - \frac{1}{2}\right)^2.$$

Expanding the terms on the right-hand side and using Lemma 3 leads to the estimate

$$D(x) = x\log x + (2\gamma - 1)x + O\left(1 + \sum_{1\leqslant s\leqslant\sqrt{x}} \psi\left(\frac{x}{s}\right)\right) \tag{1.6}$$

after a short calculation. Estimating the O-term trivially yields the desired result. □

1.5. **Relation to exponential sums.** A refinement of the estimate (1.5) for the error term in the Dirichlet divisor problem hinges upon a non-trivial estimate of the O-term in (1.6). Note that the sawtooth function $\psi(y)$ is periodic and 0 on average. Therefore, we may expect considerable cancellations in the sum

$$\sum_{1\leqslant s\leqslant\sqrt{x}} \psi\left(\frac{x}{s}\right).$$

In fact, many error terms occuring in analytic number theory can be related to sums of the form

$$\sum_{n\in I} \psi\left(g(n)\right),$$

where I is an interval and g is a "nice" function, i.e. differentiable sufficiently often and/or satisfying other good properties. We now want to relate such sums to exponential sums. The most naive approach to do this is to expand the periodic function $\psi(y)$ into a Fourier series. It turns out that

$$\psi(y) = \frac{1}{2\pi i} \cdot \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0}} \frac{e(ky)}{k} \quad \text{if } y \notin \mathbb{Z}.$$

To make this idea useful, we need to cut the summation and estimate the error term. Indeed, it can be shown that

$$\psi(y) = \frac{1}{2\pi i} \cdot \sum_{\substack{|k|\leqslant N \\ k\neq 0}} \frac{e(ky)}{k} + O\left(\frac{1}{N||y||}\right)$$

for all $N \geqslant 1$, where $||y||$ is the distance of y to the nearest integer. Now, re-arranging summations, it follows that
(1.7)

$$\sum_{n \in I} \psi\left(g(n)\right) = \frac{1}{2\pi i} \cdot \sum_{\substack{|k| \leqslant N \\ k \neq 0}} \frac{1}{k} \cdot \sum_{n \in I} e(kg(n)) + O\left(\frac{1}{N} \cdot \sum_{n \in I} \frac{1}{||g(n)||}\right).$$

Note that the inner sums

$$\sum_{n \in I} e(kg(n))$$

are exponential sums. However, we also need to deal with the O-term which makes it necessary to control the number of n's for which $g(n)$ is very close to an integer. To avoid this difficulty, we now state the following remarkable result by Vaaler which allows to approximate the sawtooth function nicely by trigonometric polynomials, which are expressions of the form

$$p(x) = \sum_{|m| \leqslant M} a_m e(mx).$$

Theorem 5 (Vaaler)**.** *Let $M \in \mathbb{N}$. Then there exists a trigonometrical polynomial*

$$\psi^*(x) := \sum_{1 \leqslant |m| \leqslant M} a_M(m) e(mx)$$

such that

$$a_M(m) \ll \frac{1}{m}$$

and

$$|\psi^*(x) - \psi(x)| \leqslant \sum_{|m| \leqslant M} b_M(m) e(mx),$$

where

$$b_M(m) := \frac{1}{2M+2} \cdot \left(1 - \frac{|m|}{M+1}\right).$$

Proof. See the appendix in [4]. Deferred to the tutorials. □

We note that the right-hand side of (5) is non-negative and that

$$b_M(m) \ll \frac{1}{M} \ll \frac{1}{m}. \tag{1.8}$$

Now using the triangle inequality and Theorem 5 gives

$$
\begin{aligned}
\left|\sum_{n\in I}\psi(g(n))\right| &\leqslant \left|\sum_{n\in I}\psi^*(g(n))\right| + \left|\sum_{n\in I}\psi(g(n)) - \sum_{n\in I}\psi^*(g(n))\right| \\
&\leqslant \left|\sum_{n\in I}\sum_{1\leqslant|m|\leqslant M} a_M(m)e(mg(n))\right| + \sum_{n\in I}\sum_{|m|\leqslant M} b_M(m)e(mg(n)) \\
&\leqslant \sum_{1\leqslant|m|\leqslant M}(|a_M(m)|+b_M(m))\cdot\left|\sum_{n\in I}e(mg(n))\right| + b_M(0)|I| \\
&\ll \frac{|I|}{M} + \sum_{1\leqslant m\leqslant M}\frac{1}{m}\cdot\left|\sum_{n\in I}e(mg(n))\right|.
\end{aligned}
$$

We summarize the above in the following result which should be compared to (1.7).

Lemma 6. *Let $M\in\mathbb{N}$ and $g: I\to\mathbb{R}$ a function. Then*

$$\left|\sum_{n\in I}\psi(g(n))\right| \ll \frac{|I|}{M} + \sum_{1\leqslant m\leqslant M}\frac{1}{m}\cdot\left|\sum_{n\in I}e(mg(n))\right|.$$

2. Lecture 2 - Van der Corput's bound and its application to the Dirichlet divisor problem

In this lecture, we prove the following basic but powerful bound for exponential sums due to van der Corput.

Theorem 7. *Suppose f is a real valued function with two continuous derivatives on the interval I of length $|I|\geqslant 1$. Suppose also that there are some $\lambda>0$ and $\alpha\geqslant 1$ such that*

$$\lambda \leqslant |f''(x)| \leqslant \alpha\lambda$$

on I. Then

$$\sum_{n\in I} e(f(n)) \ll \alpha|I|\lambda^{1/2} + \lambda^{-1/2}. \tag{2.1}$$

Before proving this result, we apply it to the Dirichlet divisor problem and prove the following result.

Lemma 8. *We have*

$$D(x) = x\log x + (2\gamma-1)x + \Delta(x)$$

with

$$\Delta(x) = O\left(x^{1/3}\log x\right). \tag{2.2}$$

Proof. We recall that we have to bound the sum $\sum_{1\leqslant s\leqslant\sqrt{x}} \psi(x/s)$. First, we split the summation range into dyadic intervals, getting

$$\sum_{1\leqslant s\leqslant\sqrt{x}} \psi\left(\frac{x}{s}\right) = \sum_{k=0}^{[\log_2\sqrt{x}]} \sum_{\sqrt{x}/2^{k+1}<s\leqslant\sqrt{x}/2^k} \psi\left(\frac{x}{s}\right). \tag{2.3}$$

Now it remains to estimate the sum $\sum_{y<s\leqslant 2y} \psi(x/s)$ for $1/2 \leqslant y \leqslant x^{1/2}$. We shall show that

$$\sum_{y<s\leqslant 2y} \psi\left(\frac{x}{s}\right) \leqslant x^{1/3} \tag{2.4}$$

under this condition. This is trivial if $y \leqslant x^{1/3}$. So we will assume that $x^{1/3} < y \leqslant x^{1/2}$ in the following. By Lemma 6, we have

$$\sum_{y/2<s\leqslant y} \psi\left(\frac{x}{s}\right) \ll \sum_{1\leqslant m\leqslant M} \frac{1}{m}\cdot\left|\sum_{y/2<n\leqslant y} e\left(m\cdot\frac{x}{n}\right)\right| + \frac{y}{M}.$$

Using Theorem 7, it follows that

$$\begin{aligned}\sum_{y/2<s\leqslant y} \psi\left(\frac{x}{s}\right) &\ll \sum_{1\leqslant m\leqslant M} \frac{1}{m}\cdot\left(y\cdot\left(m\cdot\frac{x}{y^3}\right)^{1/2} + \left(m\cdot\frac{x}{y^3}\right)^{-1/2}\right) + \frac{y}{M}\\ &\ll M^{1/2}x^{1/2}y^{-1/2} + y^{3/2}x^{-1/2} + yM^{-1}.\end{aligned}$$

Balancing the first and the last terms by choosing $M := [yx^{-1/3}]$ gives

$$\sum_{y<s\leqslant 2y} \psi\left(\frac{x}{s}\right) \leqslant x^{1/3} + y^{3/2}x^{-1/2}$$

which implies (2.4) if $x^{1/3} < y \leqslant x^{1/2}$. From (2.3) and (2.4), we deduce that

$$\sum_{1\leqslant s\leqslant\sqrt{x}} \psi\left(\frac{x}{s}\right) \ll x^{1/3}\log x,$$

which together with (1.6) implies the claim. □

The proof of Theorem 7 relies on the Kusmin-Landau bound (Lemma 10 below), which in turn is a generalization of the following simple bound for geometric sums.

Lemma 9. *For any interval I and $x \in \mathbb{R}$, we have*

$$\sum_{n\in I} e(nx) \ll \frac{1}{||x||}.$$

Proof. Assume without loss of generality that $I = (a, b]$, where $a, b \in \mathbb{Z}$. Then

$$\sum_{a<n\leqslant b} e(nx) = e(x)\cdot\frac{e(bx)-e(ax)}{e(x)-1} \ll \frac{1}{|e(x)-1|} = \frac{1}{|e(||x||)-1|}.$$

Applying Taylor approximation, we have

$$e(||x||) - 1 \gg ||x||.$$

This implies the claim. ∎

More generally, the following holds.

Lemma 10 (Kusmin-Landau)**.** *Assume f is continuously differentiable, f' is monotonic and $||f'(x)|| \geqslant \delta > 0$ on the interval I. Then*

$$\sum_{n \in I} e(f(n)) \ll \delta^{-1}.$$

Proof. See the proof of Theorem 2.1. in [4]. Deferred to the tutorials. ∎

Based on this bound, we now prove Theorem 7.

Proof. Since $|f''| \geqslant \lambda$ on I, it follows that f'' is positive or negative on I. In the following, we assume that f'' is positive. The other case that f'' is negative is similar.

Since f'' is positive on I, it follows that f' is monotonically increasing on this interval and hence f' is invertible on $(a, b]$ and $f'((a, b]) = (f'(a), f'(b)]$. Let $0 < \delta \leqslant 1/2$ be a parameter, to be fixed later. We divide the summation interval $(a, b] = (f')^{-1}(f'(a), f'(b)]$ into subintervals

$$A_k := (f')^{-1}((k - \delta, k + \delta] \cap (f'(a), f'(b)])$$

and

$$B_k := (f')^{-1}((k + \delta, k + 1 - \delta] \cap (f'(a), f'(b)]),$$

where k is an integer. For every k, the exponential sum, restricted to the summation interval A_k, will be estimated trivially, and the exponential sum, restricted to the summation interval B_k, will be estimated using Lemma 10. Hence, if $K := \sup_{k \in \mathbb{Z}} |A_k|$ and N is the number of integers k for which the interval A_k or the interval B_k is non-empty, then

$$\sum_{n \in I} e(f(n)) \ll \left(1 + K + \delta^{-1}\right) N.$$

We now want to estimate the quantities K and N in terms of the parameters α and λ in the lemma. We have

$$N \leqslant f'(b) - f'(a) + 1 \leqslant (b - a)\alpha\lambda + 1,$$

where the last inequality follows from the mean value theorem. The mean value theorem also implies that

$$K \leqslant \frac{2\delta}{\lambda}.$$

It follows that

$$\sum_{n \in I} e(f(n)) \ll \left(1 + \frac{\delta}{\lambda} + \delta^{-1}\right) (|I|\alpha\lambda + 1). \tag{2.5}$$

If $\lambda \geqslant 1/4$, then the claimed bound (2.1) is trivial. Otherwise, we choose

$$\delta := \lambda^{1/2}$$

in which case (2.1) follows from (2.5). $\square$

3. Lecture 3 - Weyl differencing (A process)

In the following lectures, we introduce two processes which transform a given exponential sums into new ones. This allows for improvements over the von der Corput bound. In particular, we will see that the exponent 1/3 in the O-term in (1.5) can be lowered. We note that the van der Corput bound (2.1) is non-trivial if

$$|I|^{-2} \ll \lambda \ll_\alpha 1.$$

(In applications, α usually just plays the role of a constant.) So in particular, if f'' is much larger than 1 on I, then we can so far not do better than estimating the exponential sum trivially. The Weyl differencing (or A process), which we disucss in this section, allows to lower the size of the second derivative of the amplitude function. Our starting point is the following general estimate which is in fact an easy consequence of the Cauchy-Schwarz inequality.

Lemma 11. *Suppose $\xi(n)$ is a complex-valued function such that $\xi(n) = 0$ if $n \notin I$. Then, if H is a positive integer, we have*

$$\left|\sum_n \xi(n)\right|^2 \leqslant \frac{|I|+H}{H} \cdot \sum_{|h|\leqslant H} \left(1 - \frac{|h|}{H}\right) \cdot \sum_n \overline{\xi(n)}\xi(n+h). \tag{3.1}$$

Proof. We write

$$H\sum_n \xi(n) = \sum_{k=1}^{H}\sum_n \xi(n+k) = \sum_n \sum_{k=1}^{H} \xi(n+k).$$

Using the Cauchy-Schwarz inequality, we deduce that

(3.2)

$$H^2 \cdot \left|\sum_n \xi(n)\right|^2 \leqslant \left(\sum_{\substack{n\in\mathbb{Z} \\ n+k\in I \text{ for some } k\in\{1,\ldots,H\}}} 1\right) \cdot \left(\sum_n \left|\sum_{k=1}^{H} \xi(n+k)\right|^2\right)$$

$$\leqslant (|I|+H) \cdot \sum_n \sum_{k=1}^{H}\sum_{l=1}^{H} \overline{\xi(n+k)}\xi(n+l).$$

Making the change of variables $m = n + k$ and $h = l - k$, we obtain (3.3)

$$\sum_n \sum_{k=1}^{H} \sum_{l=1}^{H} \overline{\xi(n+k)}\xi(n+l) = \sum_{|h|<H} \sum_m \overline{\xi(m)} \sum_{\substack{1\leqslant k,l\leqslant H \\ h=l-k}} \xi(m+h)$$
$$= \sum_{|h|<H} \sum_m \overline{\xi(m)}\xi(m+h) \cdot (H - |h|).$$

Combining (3.2) and (3.3), and dividing by H^2 gives (3.1). □

Now we apply the above general estimate to the situation when $\xi(n) = e(f(n))$ for $n \in I$ and $H \leqslant |I|$.

Theorem 12 (Weyl differencing or A process)**.** *Let $f : I \to \mathbb{R}$ be a function and $H \leqslant |I|$ an integer. Then*

$$\left|\sum_{n\in I} e(f(n))\right|^2 \leqslant \frac{2|I|^2}{H} + \frac{2|I|}{H} \cdot \sum_{1\leqslant|h|\leqslant H} \left|\sum_{n\in I_h} e(F_h(n))\right|,$$

where

$$I_h := (\max\{a, a-h\}, \min\{b, b-h\})$$

and

$$F_h(x) = f(x+h) - f(x).$$

Proof. Setting

$$\xi(n) := \begin{cases} e(f(n)) & \text{if } n \in I \\ 0 & \text{otherwise,} \end{cases}$$

and using $H \leqslant |I|$, we deduce the inequality

$$\left|\sum_{n\in I} e(f(n))\right|^2 \leqslant \frac{2|I|}{H} \cdot \sum_{|h|\leqslant H} \left|\sum_{n\in I_h} e(F_h(n))\right|$$

from Lemma 11. Now the claim follows by noting that the contribution of $h = 0$ equals $2|I|^2/H$ since $e(F_0(n)) = 1$. □

The trivial estimate is

$$\left|\sum_{n\in I} e(f(n))\right|^2 \leqslant |I|^2,$$

so the contribution $2|I|^2/H$ of $h = 0$ is roughly by a factor of H smaller than the trivial estimate, and hence we get a saving over the trivial estimate altogether if H is not too small and $\sum_{n\in I_h} e(F_h(n))$ is much smaller than $|I|$ for every h with $1 \leqslant |h| \leqslant H$. We note that the k-th derivative

$$F_h^{(k)}(x) = f^{(k)}(x+h) - f^{(k)}(x)$$

of $F_h(x)$ is, generically, much smaller than that of the original amplitude function $f(x)$, provided $|h|$ is not too large. So if the van der Corput bound fails for the original exponential sum because the second derivative of $f(x)$ is too large, we still have a chance that it works for $F_h(x)$. This is why the Weyl differencing is useful. We can even iterate the Weyl process to bring the derivatives of the amplitude function into a reasonable range. In the following, we will make the above arguments precise. We shall prove the following theorem by combinating Weyl differencing and the van der Corput bound.

Theorem 13. *Assume $f : I \to \mathbb{R}$ is three times continuously differentiable and*

$$\lambda \leqslant |f'''(x)| \leqslant \alpha\lambda \tag{3.4}$$

for some $\lambda > 0$ and $\alpha \geqslant 1$ with

$$\alpha^2\lambda \geqslant |I|^{-3} \tag{3.5}$$

and all $x \in I$. Then

$$\sum_{n\in I} e(f(n)) \ll \alpha^{1/3}|I|\lambda^{1/6} + \alpha^{1/6}|I|^{1/2}\lambda^{-1/6}. \tag{3.6}$$

Proof. Our starting point is the identity

$$F_h''(x) = f''(x+h) - f''(x) = \int_0^h f'''(x+y)\, dy$$

following from the fundamental theorem of calculus. Since f''' is continuous on I, we deduce that

$$|h|\lambda \leqslant |F_h''(x)| \leqslant |h|\alpha\lambda \text{ if } h \neq 0$$

from (3.4). Therefore, by Theorem 7,

$$\sum_{n\in I(h)} e(F_h(n)) \ll \alpha|I|(\lambda|h|)^{1/2} + (\lambda|h|)^{-1/2}.$$

Hence, if $H \leqslant |I|$ is an integer, Theorem 12 gives the bound

$$\begin{aligned}\left|\sum_{n\in I} e(f(n))\right|^2 &\ll \frac{|I|^2}{H} + \frac{|I|}{H}\cdot\left(\alpha|I|\lambda^{1/2}H^{3/2} + \lambda^{-1/2}H^{1/2}\right)\\ &\ll \alpha|I|^2\lambda^{1/2}H^{1/2} + |I|\lambda^{-1/2}H^{-1/2} + |I|^2H^{-1}.\end{aligned} \tag{3.7}$$

If $\alpha^2\lambda \geqslant 1$, then the above bound is trivial. Otherwise, we set

$$H := [\alpha^{-2/3}\lambda^{-1/3}],$$

balancing the first and the third terms in the last line of (3.7). We note that $H \leqslant |I|$ by the condition (3.5). Thus we get

$$\left|\sum_{n\in I} e(f(n))\right|^2 \ll \alpha^{2/3}|I|^2\lambda^{1/3} + \alpha^{1/3}|I|\lambda^{-1/3}$$

which implies the claim after taking the square root. □

We note that (3.6) is non-trivial if

$$|I|^{-3} \ll_\alpha \lambda \ll_\alpha 1.$$

As mentioned already, the Weyl differencing can be iterated. To do this, Theorem 12 is applied again to the new exponential sums $\sum_{n\in I_h} e(F_h(n))$ and then again to the resulting new exponential sums, and so on. We state without proof the following general result without a condition like (3.5) on α and λ which is obtained by iterating the Weyl differencing q times and then applying the van der Corput bound.

Theorem 14. *Let q be a positive integer. Suppose that f is a real valued function with $q+2$ continuous derivatives on I. Suppose also that for some $\lambda > 0$ and for some $\alpha \geqslant 1$,*

$$\lambda \leqslant |f^{(q+2)}(x)| \leqslant \alpha\lambda$$

on I. Let $Q := 2^q$. Then

$$\sum_{n\in I} e(f(n)) \ll |I| \left(\alpha^2\lambda\right)^{1/(4Q-2)} + |I|^{1-1/(2Q)}\alpha^{1/(2Q)} + |I|^{1-2/Q+1/Q^2}\lambda^{-1/(2Q)}.$$

Proof. This is Theorem 2.8. in [4]. □

4. Lecture 4 - B process, part 1: Poisson summation and bounds for exponential integrals

We have now seen how to change the sizes of the derivatives of the amplitude function. In this so-called A process (or Weyl differencing), the lengths of the summation intervals remained essentially unchanged (the length of I_h is $|I| - |h|$, which is, generically, of the same order of magnitude as $|I|$). Next, we introduce a new process which allows, under favourable circumstances, to transform exponential sums into shorter ones. This is called B process. The main ideas are to use the Poisson summation formula and then to evaluate the resulting Fourier integrals asymptotically.

4.1. **Poisson summation formula.** The famous Poisson summation formula states that under certain conditions, a series of the form $\sum_n g(n)$ equals its dual series $\sum_n \hat{g}(n)$, where $\hat{g}$ is the Fourier transform of the complex-valued function g, defined on the reals. We recall that the Fourier transform is defined as

$$\hat{g}(x) = \int_{-\infty}^{\infty} g(t)e(-tx)\ dx.$$

In the following, we state a theorem which gives precise conditions on g under which the Poisson summation formula holds. We don't look into its proof. There are many such theorems with different conditions on g in the literature.

Theorem 15. *Let $g(x)$ be a complex-valued function on the reals that is piecewise continuous with only finitely many discontinuities and for all real numbers a satisfies*

$$g(a) = \frac{1}{2} \cdot \left(\lim_{x \to a^-} g(x) + \lim_{x \to a^+} g(x) \right).$$

Moreover, assume that $g(x) \ll (1+|x|)^{-c}$ for some constant $c > 1$ with an absolute implied constant. Then

$$\sum_{n=-\infty}^{\infty} g(n) = \sum_{n=-\infty}^{\infty} \hat{g}(n).$$

Proof. See [1], for example. □

In particular, by setting

$$g(x) := \begin{cases} e(f(x)) & \text{if } a < x < b \\ e(f(x))/2 & \text{if } x = a, b \\ 0 & \text{otherwise,} \end{cases}$$

we obtain immediately the following result.

Corollary 1. *Let $a < b$ be integers and $f : [a, b] \to \mathbb{R}$ continuous. Then*

$$(4.1) \qquad \sum_{a<n\leqslant b} e(f(n)) = \frac{e(f(b)) - e(f(a))}{2} + \sum_{n=-\infty}^{\infty} \int_a^b e(f(x) - nx)\ dx.$$

This can be made useful if we are able to cut the summation on the right-hand side of (4.1). Indeed, it turns out that if f is "nice" (we later make this precise), then the Fourier integral

$$\int_a^b e(f(x) - nx)\ dx$$

gives a substantial contribution only if the first derivative of the amplitude function $f(x) - nx$ has a root in the interval $[a, b]$, i.e., $f'(x) = n$ has a solution in $[a, b]$. So, essentially, only n's in the interval

$$f'(a) \leqslant n \leqslant f'(b)$$

matter, provided f' is monotonic. Indeed, the following lemma holds.

Lemma 16. *Suppose f is a real valued function which has two continuous derivatives on the interval $[a, b]$. Suppose also that f' is monotonic* *in* $[a,b]$, *that* H_1 *and* H_2 *are integers such that* $H_1 < f'(x) < H_2$ *for* $a \leqslant x \leqslant b$ *and that* $H = H_2 - H_1 \geqslant 2$. *Then*

$$\sum_{a<n\leqslant b} e(f(n)) = \sum_{H_1 \leqslant h \leqslant H_2} \int_a^b e(f(x) - hx)\ dx + O(\log H).$$

Proof. See the proof of Lemma 3.5. in [4]. Deferred to the tutorials. □

4.2. **Bounds for exponential integrals.** Lemma 16 is the starting point of the B process. It will now be important to have good estimates or even asymptotic evaluations of the integrals $\int_a^b e(f(x) - hx)\ dx$. In this lecture, we provide two bounds for general exponential integrals $\int_a^b e(g(x))\ dx$. We first prove the following.

Lemma 17. *Assume* f *is twice differentiable on* $[a,b]$, f' *is monotonic on* $[a,b]$ *and* $|f'(x)| \geqslant \lambda_1 > 0$ *for all* $x \in [a,b]$. *Then*

$$\int_a^b e(f(x))\ dx \ll \frac{1}{\lambda_1}.$$

Proof. Integration by parts yields

(4.2)

$$\begin{aligned}\int_a^b e(f(x))\ dx =& \frac{1}{2\pi i} \cdot \int_a^b \frac{(2\pi i f'(x)) \cdot e(f(x))}{f'(x)} dx \\ =& \frac{e(f(b))}{2\pi i f'(b)} - \frac{e(f(a)}{2\pi i f'(a)} - \frac{1}{2\pi i} \cdot \int_a^b e(f(x)) \frac{d}{dx}\frac{1}{f'(x)} dx \\ \ll& \frac{1}{\lambda} + \int_a^b \left| \frac{d}{dx}\frac{1}{f'(x)} \right|\ dx.\end{aligned}$$

But $1/f'(x)$ is monotonic on $[a,b]$ since $f'(x)$ is, and hence

$$\frac{d}{dx}\frac{1}{f'(x)}$$

doesn't change sign on $[a,b]$, and therefore

$$\int_a^b \left| \frac{d}{dx}\frac{1}{f'(x)} \right|\ dx = \left| \int_a^b \frac{d}{dx}\frac{1}{f'(x)}\ dx \right| = \left| \frac{1}{f'(b)} - \frac{1}{f'(a)} \right| \ll \frac{1}{\lambda}.$$

□

Lemma 17 reflects the fact that if $e(f(x))$ oscillates quickly, then the integral is small. Next, we give a bound in terms of the second derivative of f.

Lemma 18. *Assume f is twice continuously differentiable on $[a,b]$ and $|f''(x)| \geqslant \lambda_2$ for all $x \in [a,b]$. Then*

$$\int_a^b e(f(x))\,dx \ll \frac{1}{\lambda_2^{1/2}}.$$

Proof. Let $\delta > 0$, to be fixed later. Define

$$E_1 := \{x \in [a,b] \;:\; |f'(x)| < \delta\}$$

and

$$E_2 := [a,b] \setminus E_1 = \{x \in [a,b] \;:\; |f'(x)| \geqslant \delta\}.$$

By our conditions of f, E_1 is an interval or empty, and E_2 is the union of at most 2 intervals. We shall estimate the integral on E_1 trivially and the integral on E_2 using Lemma 17. This gives

$$\int_{E_1} e(f(x))\,dx \ll |E_1| \quad \text{and} \quad \int_{E_2} e(f(x))\,dx \ll \frac{1}{\delta}.$$

It remains to bound $|E_1|$ and choose δ. If $E_1 = [c,d]$, then, by mean value theorem,

$$\frac{f'(d) - f'(c)}{d-c} = f''(x)$$

for some $x \in E_1$ and so

$$\left| \frac{f'(d) - f'(c)}{d-c} \right| \geqslant \lambda_2.$$

It follows that

$$\frac{2\delta}{\lambda_2} \geqslant \left| \frac{f'(d) - f'(c)}{\lambda_2} \right| \geqslant d - c = |E_1|.$$

Combining everything, we obtain the bound

$$\int_a^b e(f(x))\,dx \ll \frac{1}{\delta} + \frac{\delta}{\lambda_2}.$$

Now choosing $\delta := \lambda_2^{1/2}$, we obtain the desired result. □

The above two lemmas can be viewed as the continuous counterparts of the Kusmin-Landau and van der Corput bounds (Lemma 10 and Theorem 7) for (discrete) exponential sums. Also the proofs of Lemma 18 and Theorem 7 have some similarities. In both proofs, the interval $[a,b]$ is divided into subintervals on which we either use a trivial estimate or the basic estimates provided by the Kusmin-Landau bound or Lemma 17, respectively.

5. Lecture 5 - B process, part 2: Stationary phase and transformation of exponential sums into new exponential sums

5.1. **Stationary phase.** To make further progress, we need to evaluate the exponential integrals $\int\limits_a^b e(f(x)-hx)\ dx$ asymptotically. We recall that these integrals yield a significant contribution only if the derivative of the amplitude function $f(x)-hx$ has a root in $[a,b]$. We now want to look into the situation when $g(x)$ is twice continuously differentiable and $g'(x)$ is monotonic and has a root x_0 in $[a,b]$. We want to assume that $g'(x)$ is monotonically increasing in $[a,b]$ which implies that $g''(x)$ is positive. The other case that $g'(x)$ is monotonically decreasing is similar. We investigate the integral $\int\limits_a^b e(g(x))\ dx$ under these conditions. It shall turn out that only a small neighbourhood of x_0 gives a significant contribution to this integral. In this neighbourhood, the second derivative is close to $g''(x_0)$. Therefore, taking Lemma 18 into account, we may expect that

$$\int\limits_a^b e(g(x))\ dx \ll \frac{1}{\sqrt{g''(x_0)}}.$$

But we will do better than this, namely we will show that $\int\limits_a^b e(g(x))\ dx$ is close to $e(1/8+g(x_0))/\sqrt{g''(x_0)}$. The relevant asymptotic estimate is referred to as stationary phase. We shall give a detailed proof of this estimate for the simplest case when $g(x)=Ax^2$ with $A>0$ and $x_0=0$. It is easy to extend this result to arbitrary quadratic functions $g(x)$. Finally, we shall give heuristic arguments for a generalization of this result to arbitrary functions $g(x)$ satisfying the above and a few additional conditions. We shall not prove this generalization in detail.

For $g(x)=Ax^2$ with $A>0$ and $[a,b]=[-X,Y]$ with $X,Y>0$, we prove the following.

Lemma 19. *Let A, X and Y be positive numbers. Then*

$$\int\limits_{-X}^{Y} e\left(Ax^2\right)\ dx = \frac{e(1/8)}{\sqrt{2A}} + O\left(\frac{1}{AX}+\frac{1}{AY}\right).$$

Proof. We have

$$\int_{-X}^{Y} e\left(Ax^2\right)\ dx = \int_{-X}^{0} e\left(Ax^2\right)\ dx + \int_{0}^{Y} e\left(Ax^2\right)\ dx$$
$$= \int_{0}^{X} e\left(Ax^2\right)\ dx + \int_{0}^{Y} e\left(Ax^2\right)\ dx.$$

Therefore, it suffices to prove that

$$\int_{0}^{Z} e\left(Ax^2\right)\ dx = \frac{e(1/8)}{2\sqrt{2A}} + O\left(\frac{1}{AZ}\right)$$

if $Z > 0$. By a linear change of variables, we see that

$$\int_{0}^{Z} e\left(Ax^2\right)\ dx = \frac{1}{\sqrt{A}} \cdot \int_{0}^{\sqrt{A}Z} e\left(y^2\right)\ dy.$$

Hence, all we need to prove is that

$$\int_{0}^{W} e\left(y^2\right)\ dy = \frac{e(1/8)}{2\sqrt{2}} + O\left(\frac{1}{W}\right) \tag{5.1}$$

if $W > 0$.

At this point, we use complex analysis. By Cauchy's theorem, we have

$$\int_{0}^{W} e\left(y^2\right)\ dy = \int_{C_1} e\left(y^2\right)\ dy + \int_{C_2} e\left(y^2\right)\ dy, \tag{5.2}$$

where C_1 is the line seqment with initial point 0 and terminal point $e(1/8)W$, and C_2 is the circle segment with initial point $e(1/8)W$ and terminal point W and radius W. We may parametrize C_1 and the inverse $-C_2$ of C_2 as

$$C_1 : [0,1] \to \mathbb{C} \quad \text{such that } C_1(t) := e\left(\frac{1}{8}\right) \cdot tW$$

and

$$-C_2 : [0,1/8] \to \mathbb{C} \quad \text{such that } -C_2(\theta) := e\left(\theta\right) \cdot W.$$

So therefore,

$$\int_{C_1} e\left(y^2\right)\ dy = \int_{0}^{1} e\left(\frac{1}{8}\right) \cdot W \cdot e\left(it^2W^2\right)\ dt \tag{5.3}$$

and

$$\int_{C_2} e\left(y^2\right)\, dy = -\int_0^{1/8} 2\pi i \cdot e(\theta) \cdot W \cdot e\left(e(2\theta)W^2\right)\, d\theta. \tag{5.4}$$

The integral on the right-hand side of (5.3) turns into

$$\begin{aligned}\int_0^1 e\left(\frac{1}{8}\right) \cdot W \cdot e\left(it^2W^2\right)\, dt =& e\left(\frac{1}{8}\right) \cdot \int_0^1 W \cdot e^{-2\pi t^2 W^2}\, dt \\ =& e\left(\frac{1}{8}\right) \cdot \int_0^W e^{-2\pi z^2}\, dz\end{aligned}$$

after a linear change of variables. Further, a change $x = 2\pi z^2$ of variables gives

$$\begin{aligned}e\left(\frac{1}{8}\right) \cdot \int_0^W e^{-2\pi z^2}\, dz =& e\left(\frac{1}{8}\right) \cdot \int_0^{2\pi W^2} \frac{1}{2\sqrt{2\pi}} \cdot x^{-1/2}e^{-x}\, dx \\ =& \frac{e(1/8)}{2\sqrt{2\pi}} \cdot \left(\int_0^{\infty} x^{-1/2}e^{-x}\, dx - \int_{2\pi W^2}^{\infty} x^{-1/2}e^{-x}\, dx\right).\end{aligned}$$

We remember that

$$\int_0^{\infty} x^{-1/2}e^{-x} = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

and estimate the second integral by

$$\int_{2\pi W^2}^{\infty} x^{-1/2}e^{-x}\, dx \leqslant \int_0^{\infty} (2\pi W^2)^{-1/2}e^{-x}\, dx \ll \frac{1}{W}.$$

So altogether, we obtain the asymptotic estimate

$$\int_{C_1} e\left(y^2\right)\, dy = \frac{e(1/8)}{2\sqrt{2}} + O\left(\frac{1}{W}\right) \tag{5.5}$$

for the integral on C_1.

For the integral on the right-hand side of (5.4), we see that

$$\begin{aligned}
&-\int_0^{1/8} 2\pi i \cdot e(\theta) \cdot W \cdot e\left(e(2\theta)W^2\right)\, d\theta \\
=&-2\pi i W \int_0^{1/8} e(\theta) \cdot e^{2\pi i(\cos(4\pi\theta)+i\sin(4\pi\theta))W^2}\, d\theta \\
=&-2\pi i W \int_0^{1/8} e(\theta) \cdot e^{2\pi i \cos(4\pi\theta)W^2} \cdot e^{-2\pi \sin(4\pi\theta)W^2}\, d\theta \\
\ll& W \int_0^{1/8} e^{-2\pi \sin(4\pi\theta)W^2}\, d\theta \\
\leqslant& W \int_0^{1/8} e^{-4\pi^2\theta W^2}\, d\theta \\
=&\frac{W}{4\pi^2 W^2} \cdot \left(1-e^{-\pi^2 W^2/2}\right)\, d\theta \\
\ll&\frac{1}{W}
\end{aligned}$$

and hence

$$\int_{C_2} e\left(y^2\right)\, dy = O\left(\frac{1}{W}\right). \tag{5.6}$$

Combining (5.2), (5.5) and (5.6), we obtain (5.1). □

From the above lemma, we deduce the following more general result by a change of variables.

Corollary 2. *Let a, b, x_0 be real numbers such that $a < x_0 < b$ and $A > 0$ and B be real numbers. Then*

$$\int_a^b e\left(B + A(x-x_0)^2\right)\, dx = \frac{e(1/8+B)}{\sqrt{2A}} + O\left(\frac{1}{A(x_0-a)} + \frac{1}{A(b-x_0)}\right).$$

This fails to be an asymptotic estimate if x_0 is very close to one of the endpoints a or b of the interval $[a, b]$. However, thanks to Lemma 18, we know that

$$\int_a^b e\left(B + A(x-x_0)^2\right)\, dx = O\left(\frac{1}{\sqrt{A}}\right),$$

and hence we deduce the following improved version from Corollary 2 with an error term of the same size as the main term if x_0 is very close to a or b (i.e., in this situation, we just have the above upper bound).

Corollary 3. *Let a, b, x_0 be real numbers such that $a < x_0 < b$ and $A > 0$ and B be real numbers. Then*

$$\int_a^b e\left(B + A(x - x_0)^2\right)\, dx$$

$$= \frac{e(1/8 + B)}{\sqrt{2A}} + O\left(\min\left(\frac{1}{A(x_0 - a)}, \frac{1}{\sqrt{A}}\right) + \min\left(\frac{1}{A(b - x_0)}, \frac{1}{\sqrt{A}}\right)\right).$$

Now we want to extend this result to general amplitude functions $g(x)$. We give a heuristic argument before stating the precise result. Assume that $g(x)$ is twice continuously differentiable in $[a, b]$ and $g'(x)$ is monotonically increasing in $[a, b]$ (which implies that $g''(x)$ is positive in $[a, b]$). Assume that $g'(x_0) = 0$ for some $x_0 \in [a, b]$. Then the second order Taylor approximation of $g(x)$ at x_0 takes the form

$$g(x) = g(x_0) + \frac{1}{2} \cdot g''(x_0) \cdot (x - x_0)^2 + \text{ error term.}$$

So taking Corollary 3 into account, we may expect to get a result of the shape

$$\int_a^b e(g(x))\, dx = \frac{e(1/8 + g(x_0))}{g''(x_0)^{1/2}}$$

$$+ O\Big(\min\left\{\frac{1}{g''(x_0)(x_0 - a)}, \frac{1}{g''(x_0)^{1/2}}\right\} + \min\left\{\frac{1}{g''(x_0)(b - x_0)}, \frac{1}{g''(x_0)^{1/2}}\right\}$$

$$+ \text{ further error terms.}\Big)$$

The precise result is stated below. Its proof is based on the above idea but very technical and therefore we omit it here. Some conditions on the third and fourth derivatives of $g(x)$ are required in addition.

Theorem 20 (Stationary Phase)**.** *Suppose g is a real valued function with four continuous derivatives on $[a, b]$. Suppose also that $g''(x) \geqslant \lambda_2 > 0$ on $[a, b]$ and $g'(x_0) = 0$ for some $x_0 \in [a, b]$. Finally, assume that*

$$|g^{(3)}(x)| \leqslant \lambda_3 \quad \text{and} \quad |g^{(4)}(x)| \leqslant \lambda_4$$

on $[a, b]$. Then

$$\int_a^b e(g(x))\, dx = \frac{e(1/8 + g(x_0))}{g''(x_0)^{1/2}} + O\left(R_1 + R_2\right),$$

where

$$R_1 := \min\left(\frac{1}{\lambda_2(x_0-a)}, \frac{1}{\lambda_2^{1/2}}\right) + \min\left(\frac{1}{\lambda_2(b-x_0)}, \frac{1}{\lambda_2^{1/2}}\right)$$

and

$$R_2 = (b-a)\lambda_4\lambda_2^{-2} + (b-a)\lambda_3^2\lambda_2^{-3}.$$

Proof. This is Lemma 3.6. in [4]. □

5.2. **Transformation of exponential sums into new exponential sums.** Now we are ready to formulate the B process.

Theorem 21 (B process)**.** *Suppose that f has four continuous derivatives on $[a,b]$, and that $f''(x) > 0$ on this interval. Suppose further that $[a,b] \subseteq [N, 2N]$ and that $\alpha = f'(a)$ and $\beta = f'(b)$. Assume that there is some $F > 0$ such that*

$$f''(x) \asymp FN^{-2}, \quad f^{(3)}(x) \ll FN^{-3}, \quad f^{(4)}(x) \ll FN^{-4}$$

on $[a,b]$. Let x_h be defined by the relation $f'(x_h) = h$, and let $\phi(h) = f(x_h) - hx_h$. Then

(5.7)

$$\sum_{a<n\leqslant b} e(f(n)) = \sum_{\alpha\leqslant h\leqslant\beta} \frac{e(1/8+\phi(h))}{f''(x_h)^{1/2}} + O\left(\log(FN^{-1}+2) + F^{-1/2}N\right).$$

Proof. We just sketch how it works. The details are left to the reader as an exercise. First, by Lemma 16, we have

$$\sum_{a<n\leqslant b} e(f(n)) = \sum_{[\alpha]-1\leqslant h\leqslant[\beta]+1} \int_a^b e(f(x)-hx)\,dx + O\left(\log(2+\beta-\alpha)\right)$$

with $\alpha := f'(a)$ and $\beta := f'(b)$. Now, if $h \in [\alpha, \beta]$, then there is a unique solution x_h of the equation $(f(x)-hx)' = 0$ (or $f'(x) = h$) in $[a,b]$. In this situation, we use Theorem 20 to approximate the integral

$$\int_a^b e(f(x)-hx)$$

by

$$\frac{e(1/8+f(x_h)-hx_h)}{f''(x_h)^{1/2}}.$$

Otherwise, $h = [\alpha]-1$, $[\alpha]$ or $[\beta]+1$, in which situation we bound the said integral using Lemma 18 by $1/\lambda_2^{1/2}$. Doing this, we end up with an expression of the form

$$\sum_{a<n\leqslant b} e(f(n)) = \sum_{\alpha\leqslant h\leqslant\beta} \frac{e(1/8+f(x_h)-hx_h)}{f''(x_h)^{1/2}} + \text{ sum of error terms.}$$

It remains to bound the sum of error terms which can be easily done under the conditions in this Theorem. □

We note that the factor $1/f''(x_h)^{1/2}$ on the right-hand side of (5.7) can be removed using partial summation. In this way, one obtains the bound

$$\sum_{\alpha\leqslant h\leqslant\beta} \frac{e(1/8+\phi(h))}{f''(x_h)^{1/2}} \ll \frac{N}{F^{1/2}}\cdot \sup_{\alpha\leqslant x\leqslant\beta}\left|\sum_{\alpha\leqslant h\leqslant x} e(\phi(h))\right|. \tag{5.8}$$

Thus, we have successfully transformed exponential sums into new ones. Similarly to Theorem 21, one can prove the following for the case when $f''(x)<0$ on $[a,b]$.

Theorem 22 (B process, variant)**.** *Suppose that f has four continuous derivatives on $[a,b]$, and that $f''(x)<0$ on this interval. Suppose further that $[a,b]\subseteq[N,2N]$ and that $\alpha=f'(b)$ and $\beta=f'(a)$. Assume that there is some $F>0$ such that*

$$f''(x)\asymp FN^{-2},\quad f^{(3)}(x)\ll FN^{-3},\quad f^{(4)}(x)\ll FN^{-4}$$

on $[a,b]$. Let x_h be defined by the relation $f'(x_h)=h$, and let $\phi(h)=f(x_h)-hx_h$. Then

$$\sum_{a<n\leqslant b} e(f(n)) = \sum_{\alpha\leqslant h\leqslant\beta}\frac{e(-1/8+\phi(h))}{|f''(x_h)|^{1/2}} + O\left(\log(FN^{-1}+2)+F^{-1/2}N\right). \tag{5.9}$$

Similarly to (5.8), we have

$$\sum_{\alpha\leqslant h\leqslant\beta} \frac{e(-1/8+\phi(h))}{|f''(x_h)|^{1/2}} \ll \frac{N}{F^{1/2}}\cdot \sup_{\alpha\leqslant x\leqslant\beta}\left|\sum_{\alpha\leqslant h\leqslant x} e(\phi(h))\right|. \tag{5.10}$$

It is advantageous to turn from the exponential sum $\sum\limits_{a<n\leqslant b} e(f(n))$ to the new exponential sums $\sum\limits_{\alpha<h\leqslant x} e(\phi(h))$ if the summation range gets shorter. In the generic case, we have $b-a\asymp N$ and $x-\alpha\asymp\beta-\alpha\asymp FN^{-1}$. Thus as a rule of thumb, the B process is advantageous if $F\leqslant N^2$ (unless $b-a$ is small compared to N).

6. Lecture 6 - Exponent pairs and applications

6.1. **Exponent pairs.** Now we want to combine the A and B processes developed in the previous lectures to obtain improved estimates for exponential sums. A very general framework for this is the theory of exponent pairs. We describe it heuristically without going into the highly technical details, which can be found in the relevant literature (e.g. [4]).

Assume in the following that $I=(a,b]\subseteq(N,2N]$ and f is a function on $[a,b]$ which satisfies the conditions in Theorems 21 or 22. These conditions are satisfied in most applications. Assume, in addition, that

$$f'(x)\asymp FN^{-1}=:L$$

on I, which is also the case in most applications. Prototypes are the functions $f(x)=yx^s$ with $y\neq 0$ and $s\notin\{0,1,2\}$. Let us re-inspect van der Corput's bound, Theorem 7. Under the conditions above, we have

(6.1)

$$\begin{aligned}\sum_{n\in I}e(f(n))\ll&N\cdot\left(\frac{F}{N^2}\right)^{1/2}+\left(\frac{F}{N^2}\right)^{-1/2}\\ =&F^{1/2}+\frac{N}{F^{1/2}}=L^{1/2}N^{1/2}+L^{-1/2}N^{1/2}\ll L^{1/2}N^{1/2}\end{aligned}$$

if $L\geqslant 1$. Note the exponent pair $(1/2,1/2)$ above. By a sequence of A and B processes, it is possible to produce new exponent pairs (k,l), i.e. estimates of the form

$$\sum_{n\in I}e(f(n))\ll L^kN^l,$$

as we will indicate below. The conditions on f are highly technical, though, so we will not state them here. Prototypes are functions f such that $f'(x)=yx^{-t}$ with $y\neq 0$ and $t>0$. More precisely, given any exponent pair (k,l), we can produce new ones, $A(k,l)=(k',l')$ and $B(k,l)=(k'',l'')$, using the A and B process, respectively.

Roughly, this works as follows. Recall Theorem 12. Using the mean value theorem, we have

$$F_h'(n)=f'(n+h)-f'(n)=hf''(x)\asymp|h|FN^{-2}=|h|LN^{-1}$$

for some $x\in[n,n+h]$. Hence, if (k,l) is an exponent pair, then

$$\sum_{n\in I_h}e(F_h(n))\ll\left(|h|LN^{-1}\right)^kN^l=|h|^kL^kN^{l-k}.$$

Using Theorem 12, it follows that

$$\begin{aligned}\sum_{n\in I}e(f(n))\ll&\frac{N}{H^{1/2}}+\frac{N^{1/2}}{H^{1/2}}\cdot\left(\sum_{1\leqslant|h|\leqslant H}|h|^kL^kN^{l-k}\right)^{1/2}\\ \ll&NH^{-1/2}+N^{(l-k+1)/2}H^{k/2}L^{k/2}.\end{aligned}$$

We balance the two terms in the last bound choosing

$$H:=\left[L^{-k/(k+1)}N^{(k-l+1)/(k+1)}\right],$$

thus getting

$$\sum_{n\in I}e(f(n))\ll L^{k/(2(k+1))}N^{(k+l+1)/(2(k+1))}.$$

So if (k,l) is an exponent pair, then so is

$$A(k,l) := \left(\frac{k}{2(k+1)}, \frac{k+l+1}{2(k+1)}\right). \tag{6.2}$$

Next, recall Theorem 21 (resp. Theorem 22) and (5.8) (resp. (5.10)). Consider, for the moment, h as a continuous variable. Then, using simple calculus and $f'(x_h) = h$, which is equivalent to $x_h = (f')^{-1}(h)$, we have

$$\Phi'(h) = f'(x_h)\cdot\frac{d}{dh}x_h - x_h - h\cdot\frac{d}{dh}x_h = \frac{f'(x_h)}{f''(x_h)} - x_h - \frac{h}{f''(x_h)} = -x_h \asymp N.$$

Hence, if (k,l) is an exponent pair, then, using $\alpha, x \sim L$, it follows that

$$\sum_{\alpha\leqslant h\leqslant x} e(\phi(h)) \ll N^k L^l.$$

Therefore, using Theorem 21 (resp. Theorem 22) and (5.8) (resp. (5.10)), we obtain

$$\sum_{a<n\leqslant b} e(f(n)) \ll \frac{N^k L^l}{(L/N)^{1/2}} + \text{ small errors} = L^{l-1/2}N^{k+1/2} + \text{ small errors}.$$

So under suitable conditions on f, if (k,l) is an exponent pair, then so is

$$B(k,l) := (l-1/2, k+1/2). \tag{6.3}$$

We see that $B^2(k,l) = B(B(k,l)) = (k,l)$. So two consecutive B-processes take us back to where we came from. This is no surprise but just reflects the fact that applying the Poisson summation formula (which is underlying the B process) twice brings us back to the original sum since

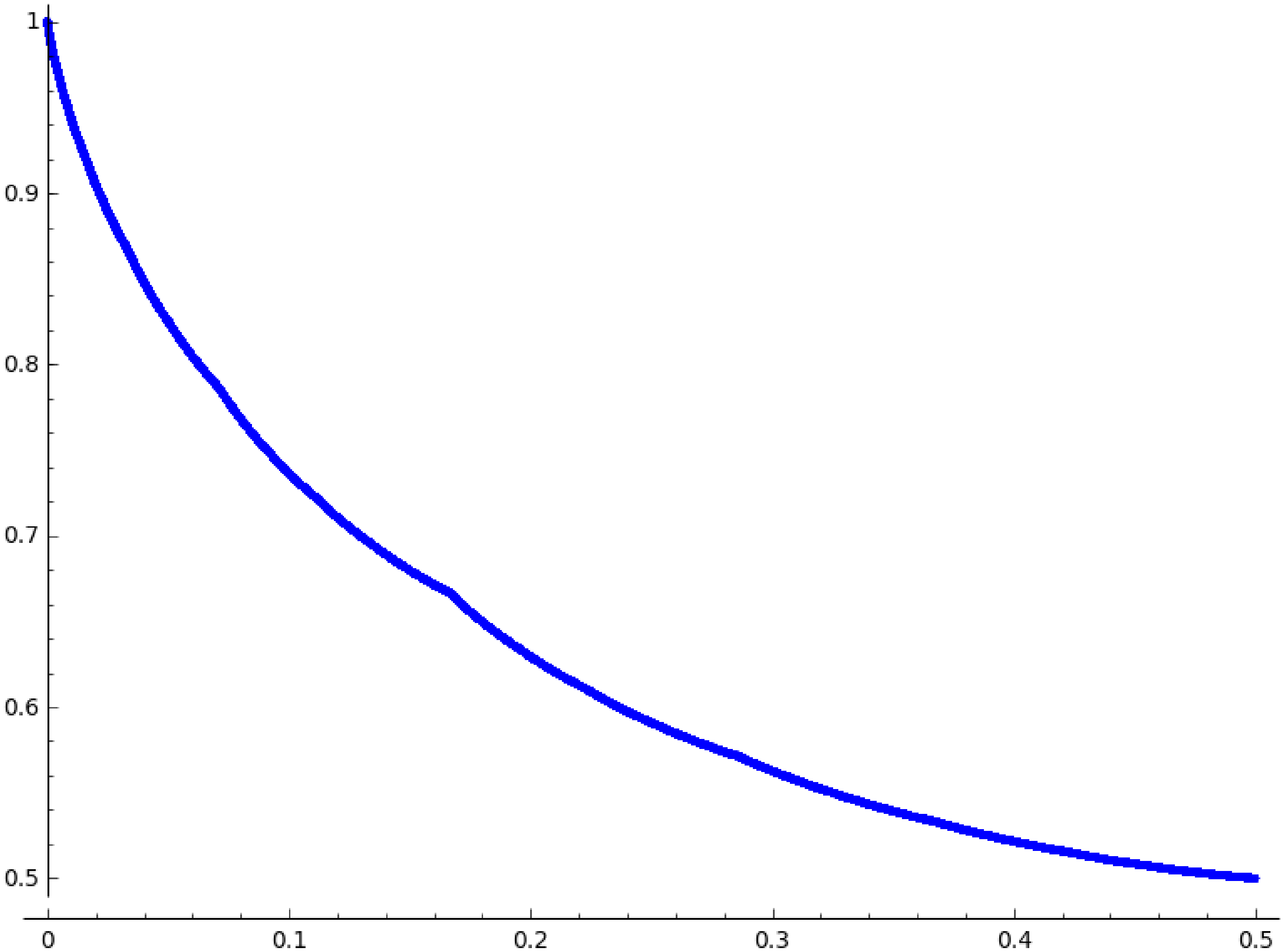
$$\sum_{n\in\mathbb{Z}} g(n) = \sum_{n\in\mathbb{Z}} \hat{g}(n) = \sum_{n\in\mathbb{Z}} \hat{\hat{g}}(n) = \sum_{n\in\mathbb{Z}} g(-n) = \sum_{n} g(n).$$

The simplest exponent pair, coming from the trivial bound, is $(0,1)$. We have $A(0,1) = (0,1)$, so applying A process doesn't help here. But applying B process gives $B(0,1) = (1/2, 1/2)$, which corresponds to the van der Corput bound, Theorem 7, as we see from our computations at the beginning of this subsection. Now applying A process gives $(1/6, 2/3)$, which corresponds to Theorem 3.7. More generally, applying A^q to $(1/2,1/2)$ corresponds to Theorem 14.

The set $\mathcal{E}$ of all exponent pairs (k,l) that we can produce by iterating A and B processes consists of the pairs of reals

$$B^{\epsilon}A^{q_k}B...A^{q_3}BA^{q_2}BA^{q_1}B(0,1) = 1,$$

where $(q_1, q_2, q_3, ..., q_k)$ is any finite sequence of non-negative integers and $\epsilon = 0,1$. It can be seen that always $0 \leqslant k \leqslant 1/2 \leqslant l \leqslant 1$. The set $\mathcal{E}$ is interesting. It forms a curve with cusps at infinitely many points. Below a picture, produced by a computer program written by Kamalakshya Mahatab (IMSc Chennai).



Bombieri and Iwaniec [2] produced new (and smaller) exponent pairs by an ingenious method going beyond applying A and B processes (sketched in lecture 7). In particular, they proved that $(9/56+\varepsilon, 37/56+\varepsilon)$ is an exponent pair. Huxley and Watt later refined their method, producing slightly smaller exponent pairs. It is conjectured that $(\varepsilon, 1/2+\varepsilon)$ is an exponent pair for every $\varepsilon > 0$, i.e. for a certain (large) class of functions f, we should have

$$\sum_{n\in I} e(f(n)) = N^{1/2+\varepsilon}.$$

This so-called "exponent pair hypothesis" would imply the Lindelöf hypothesis and also the optimal bound for the error term in the Dirichlet divisor problem, as we will see in the following subsection.

Let us look again at the prototype functions $f(x)$ for which $f'(x) = yx^{-t}$ with $y > 0$ and $t > 0$. In this case, the exponent pair (k, l) gives

$$\sum_{n\in I} e(f(n)) \ll \left(yN^{-t}\right)^k N^l = y^k N^{l-tk}.$$

If, in addition, the size of y depends on that of N, $y \asymp N^u$, say, then we deduce that

$$\sum_{n\in I} e(f(n)) \ll N^{(u-t)k+l}.$$

To get an as good as possible bound, we thus need to minimize $(u-t)k + l$. A general algorithm to minimize the quantity

$$\frac{ak + bl + c}{dk + el + f}$$

for $(k, l) \in \mathcal{E}$ and given reals a, b, c, d, e, f is described in [4].

6.2. **Applications.** We return to the Dirichlet divisor problem and try to improve upon the exponent 1/3 in the error term bound (2.2) in Lemma 8. We will be a bit sloppy since we haven't developed the theory of exponent pairs in full detail. Recall the proof of the said lemma. So we aim for a bound of the form

$$\sum_{y<s\leqslant 2y} \psi\left(\frac{x}{s}\right) \leqslant x^w$$

with $w < 1/3$ if $1 \leqslant y \leqslant x^{1/2}$. If $y \leqslant x^w$, then this is trivial, so we may assume that $x^w < y \leqslant x^{1/2}$. Recall the inequality

$$\sum_{y/2<s\leqslant y} \psi\left(\frac{x}{s}\right) \ll \sum_{1\leqslant m\leqslant M} \frac{1}{m} \cdot \left| \sum_{y/2<n\leqslant y} e\left(m \cdot \frac{x}{n}\right)\right| + \frac{y}{M} \tag{6.4}$$

from the said proof. So if (k, l) is an exponent pair, then (after properly checking the conditions on the function $f(n) = mx/n$ from the theory of exponent pairs) we will get

$$\sum_{y/2<n\leqslant y} e\left(m \cdot \frac{x}{n}\right) \ll \left(\frac{mx}{y^2}\right)^k y^l = (mx)^k y^{l-2k}.$$

Hence, from (6.4), we deduce that

$$\sum_{y/2<s\leqslant y} \psi\left(\frac{x}{s}\right) \ll (Mx)^k y^{l-2k} + \frac{y}{M},$$

provided $k > 0$. Choosing

$$M := \left[x^{-k/(k+1)} y^{(1+2k-l)/(k+1)}\right],$$

we thus obtain

$$\sum_{y/2<s\leqslant y} \psi\left(\frac{x}{s}\right) \ll y^{(l-k)/(k+1)} x^{k/(k+1)}.$$

We recall that $k \leqslant l$. So since $y \leqslant x^{1/2}$, it follows that

$$\sum_{y/2<s\leqslant y} \psi\left(\frac{x}{s}\right) \ll x^{(k+l)/(2(k+1))}.$$

So by the first steps in the proof of Lemma 8, we deduce that the exponent 1/3 in (2.2) can be replaced by $D(k,l) := (k+l)/(2(k+1))$.

Note that $(k,l) = (1/2,1/2) = B(0,1)$ gives us back $D(k,l) = 1/3$. So what happens if we apply an A process after the B process? Then we get, as already mentioned, the exponent pair $AB(0,1) = (1/6,2/3)$, yielding $D(k,l) = 5/14 > 1/3$. This worsens the result. So we need to apply another A process since applying B process fixes this (and only this) exponent pair $(1/6,2/3)$ because $B(1/6,2/3) = (1/6,2/3)$ by (6.3). We compute that $A^2B(0,1) = A(1/6,2/3) = (1/14,11/14)$. This gives $D(k,l) = 2/5$ which is even worse. Applying one more B process, we get $BA^2B(0,1) = B(1/14,11/14) = (2/7,4/7)$, which gives $D(k,l) = 1/3$. So we are back to the original exponent. Applying one more A process will again worsen the result, so let's try $BA^3B(0,1) = BA(1/14,11/14) = B(1/30,26/30) = (11/30,16/30)$. This gives $D(k,l) = 27/82 < 1/3$. Finally, we have improved upon 1/3! This exponent 27/82 was obtained by van der Corput already in 1927.

The general algorithm, mentioned earlier, gives us a complicated sequence, leading to $D(k,l) = 0.329021...$. Under the exponent pair hypothesis, i.e. the conjecture that $(\varepsilon, 1/2+\varepsilon)$ is an exponent pair for every $\varepsilon > 0$, we would get the optimal exponent $D(k,l) = 1/4+\varepsilon'$ (where ε' depends on ε and can be made arbitrarily small by making ε small enough). The record so far is $131/416 = 0.31490...$ due to Huxley [7]. His method refines that of Bombieri-Iwaniec [2]. Let us state his result as a theorem.

Theorem 23 (Huxley, 2003)**.** *We have*

$$D(x) = x\log x + (2\gamma - 1)x + \Delta(x)$$

with

$$\Delta(x) = O\left(x^{131/416+\varepsilon}\right).$$

Now that we have dealt with the Dirichlet divisor problem again, let us briefly look at another of the numerous applications of the theory of exponent pairs. The growth of the Riemann zeta function on the critical line is a very important problem. Under the Riemann Hypothesis, we know that the Lindelöf hypothesis,

$$\zeta\left(\frac{1}{2}+it\right) \ll_\varepsilon |t|^\varepsilon \quad \text{for every } \varepsilon > 0 \text{ and } |t| \geqslant 1$$

holds. The convexity bound, which just follows from the Phragmen-Lindelöf principle and the functional equation for the Riemann zeta function, is

$$\zeta\left(\frac{1}{2}+it\right) \ll |t|^{1/4}.$$

The way to link the Riemann zeta function on the critical line with exponential sums is to approximate $\zeta\left(\frac{1}{2}+it\right)$ by expressions of the shape

$$\sum_{n=1}^{T} n^{-1/2-it}.$$

The series

$$\zeta(s)=\sum_{n=1}^{\infty} n^{-s}$$

defining $\zeta(s)$ for $\Re s > 1$ doesn't necessarily converge if $\Re s < 1$, but partial sums of this series give good approximations. Now, the above sum can be rewritten as weighted exponential sum, namely

$$\sum_{n=1}^{T} n^{-1/2-it} = \sum_{n=1}^{T} n^{-1/2}\cdot e\left(\frac{t}{2\pi}\cdot\log n\right),$$

and the weight $n^{1/2}$ can be removed using partial summation. As an outcome of this, one can show that

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \ll (\log|t|)\cdot\left(1+\sup_{1\leqslant y\leqslant z\leqslant 2y\leqslant t} y^{-1/2}\left|\sum_{y<n\leqslant z} e\left(\frac{t}{2\pi}\cdot\log n\right)\right|\right).$$

Now, exponent pairs give

$$\sum_{y<n\leqslant z} e\left(\frac{t}{2\pi}\cdot\log n\right) \ll \left(\frac{t}{y}\right)^{k} y^{l} \ll |t|^{k} y^{l-k}.$$

Since $(l-1/2, k+1/2)=B(k,l)$ is another exponent pair, we also have

$$\sum_{y<n\leqslant z} e\left(\frac{t}{2\pi}\cdot\log n\right) \ll |t|^{l-1/2} y^{k-l+1}.$$

Taking the geometric mean of these two bounds, we get

$$\sum_{y<n\leqslant z} e\left(\frac{t}{2\pi}\cdot\log n\right) \ll |t|^{(2k+2l-1)/4} y^{1/2}$$

which implies

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \ll |t|^{(2k+2l-1)/4}\log|t|. \tag{6.5}$$

Thus, we need to minimize the quantity $Z(k,l)=(2k+2l-1)/4$. The general algorithm leads to $Z(2k+2l-1)/4 = 0.16451\ldots$. The exponent pair hypotesis would give $Z(k,l)=\varepsilon$ and hence the Lindelöf hypothesis. The latest record, using methods going beyond the classical theory of exponent pairs, due to Bourgain [3]. Bourgain's method improves that of Huxley [7] at a particular point. Let us state his result as a theorem.

Theorem 24 (Bourgain, 2016)**.** *For every* $\varepsilon > 0$ *and* $|t| \geqslant 1$*, we have*

$$\zeta\left(\frac{1}{2} + it\right) \ll_\varepsilon |t|^{13/84+\varepsilon}.$$

7. Lecture 7 - The method of Bombieri and Iwaniec

In a nutshell, Bombieri's and Iwaniec's method, developed in [2], works as follows. After an averaging process with some similarity to Weyl differencing, one is left with sums of short exponential sums with amplitude function $G_m(n) = f(m+n) - f(m)$. This function is developed into a cubic Taylor polynomial. The coefficients of the linear and quadratic terms are approximated by rationals with the same denominator. One is left with sums of incomplete quadratic Gauss sums, disturbed by a cubic term. After applying Poisson summation, this sum is turned into a sum of products of complete quadratic Gauss sums and cubic exponential integrals. The quadratic Gauss sums are evaluated explicitly, and the cubic exponential integrals are approximated via stationary phase. Combining everything, one is led to sums of exponential terms $e(g)$, g depending on several parameters. These sums are treated using the double large sieve. This leads to a complicated spacing problem regarding the points g.

In the following, we give a more detailed description, but the exposition remains very sketchy. We follow [4], section 7. Bombieri and Iwaniec [2] established their method for the special function $f(n) = t\log n$ to deduce a new bound for the growth of the Riemann zeta function, namely

$$\zeta\left(\frac{1}{2} + it\right) \ll t^{9/56+\varepsilon}. \tag{7.1}$$

Their method has been extended by Huxley and Watt [5] to general amplitude functions $f(n)$ in the class to which the theory of exponent pairs applies. The new exponent pair produced by this method is the following.

Theorem 25. *For every* $\varepsilon > 0$*,*

$$\left(\frac{9}{56} + \varepsilon, \frac{37}{56} + \varepsilon\right)$$

is an exponent pair.

Note that due to (6.5), the above implies (7.1). We have seen that $AB(0,1) = (1/6, 2/3)$ is an exponent pair. Note that $9/56 < 1/6$ and $37/56 < 2/3$. So we really get a genuinly new exponent pair which cannot be obtained using only A and B processes.

7.1. **Averaging process.** We consider exponential sums over dyadic intervals,

$$S = \sum_{M<m\leqslant M_1} e(f(m))$$

with M, M_1 integers such that $M < M_1 \leqslant 2M$. Let N be an integer, to be fixed later, which is thought of as small compared to M but not too small. Suppose that $N \leqslant (M_1 - M)/2$. We write

$$NS = \sum_{N<n\leqslant 2N} \sum_{M<m\leqslant M_2} e(f(m+n)) + O\left(N^2\right),$$

where $M_2 := M_1 - 2N$. It follows that

$$S \ll \frac{1}{N} \cdot \sum_{M<m\leqslant M_2} \left| \sum_{N<n\leqslant 2N} e(f(m+n) - f(m)) \right| + O(N).$$

Thus, we have estimated S in terms of a sum of short sums.

7.2. **Approximation by a cubic Taylor polynomial.** Now, for $M < m \leqslant M_2$ and $N < n \leqslant 2N$, we write

$$f(m+n) - f(m) = f'(m)n + \frac{f''(m)}{2} \cdot n^2 + \frac{f'''(m)}{6} \cdot n^3 + \text{ error.}$$

The error is small and can be removed if f satisfies suitable conditions and N is small enough. It remains to bound

$$\sum_{M<m\leqslant M_2} \left| \sum_{N<n\leqslant N_0} e\left(f'(m)n + \frac{f''(m)}{2} \cdot n^2 + \frac{f'''(m)}{6} \cdot n^3\right) \right|$$

for $N < N_0 \leqslant 2N$.

7.3. **Diophantine approximation.** Let $M < m \leqslant M_2$. By Dirichlet's approximation theorem, there exist coprime integers a, c with $1 \leqslant c \leqslant N$ such that the middle coefficent of the above cubic polynomial satisfies

$$\left| \frac{f''(m)}{2} - \frac{a}{c} \right| \leqslant \frac{1}{cN}.$$

We write

$$m = [m(a,c)] + l, \tag{7.2}$$

where $m(a,c)$ is the solution x of

$$\frac{f''(x)}{2} = \frac{a}{c},$$

where we impose the condition that f'' is monotonic on $[M, M_1]$. Thus our sum in question turns into

$$\sum_c \sum_a \sum_l \left| \sum_{N<n\leqslant N_0} e\left(f'(m)n + \frac{f''(m)}{2} \cdot n^2 + \frac{f'''(m)}{6} \cdot n^3\right) \right|,$$

where c, a, l are in suitable ranges and m depends on c, a, l by the equation (7.2). We replace the coefficient $f''(m)/2$ by a/c and the coefficient $f'''(m)/6$ by

$$\mu = \frac{1}{6} \cdot f'''(m(a,c))$$

at the cost of a small error which can be removed. This leads us to sums of the form

$$\sum_c \sum_a \sum_l \left| \sum_{N<n\leqslant N_1} e\left(f'(m)n + \frac{a}{c} \cdot n^2 + \mu n^3\right) \right|$$

with $N < N_1 \leqslant N_0$.

Next, we approximate the coefficient $f'(m)$ in terms of c, a, l. By Taylor expansion and the definitions of $m(a,c)$ and l, we have

$$\begin{aligned} f'(m) =& f'([m(a,c)]) + lf''([m(a,c)]) + \text{ error} \\ =& f'([m(a,c)]) + lf''(m(a,c)) + \text{ error}' = f'([m(a,c)]) + \frac{2la}{c} + \text{ error}'. \end{aligned}$$

Further, let b be an integer such that b/c is as close to $f'(m(a,c))$ as possible. Then, altogether, we get an approximation of $f'(m)$ by

$$f'(m) = \frac{b + 2al}{c} + \text{ error}.$$

Eventually, we are led to sums of the form

$$\sum_c \sum_a \sum_l \left| \sum_{N<n\leqslant N_2} e\left(\frac{b+2al}{c} \cdot n + \frac{a}{c} \cdot n^2 + \mu \cdot n^3\right) \right|$$

with $N < N_2 \leqslant N_1$, where b and μ depend on c, a, l. (In fact, the story is slightly more complicated, but we cheat a bit.)

7.4. **Application of Poisson summation.** We now transform the incomplete quadratic Gauss sum with cubic perturbance,

$$\sum_{N<n\leqslant N_2} e\left(\frac{b+2al}{c} \cdot n + \frac{a}{c} \cdot n^2 + \mu \cdot n^3\right)$$

using Poisson summation. First, we split the summation over n into residue classes modulo c, getting

$$\begin{aligned} &\sum_{N<n\leqslant N_2} e\left(\frac{b+2al}{c} \cdot n + \frac{a}{c} \cdot n^2 + \mu \cdot n^3\right) \\ =& \sum_{d=1}^{c} e\left(\frac{b+2al}{c} \cdot d + \frac{a}{c} \cdot d^2\right) \cdot \sum_{\substack{N<n\leqslant N_2 \\ n\equiv d \bmod c}} e\left(\mu \cdot n^3\right). \end{aligned}$$

We make a linear change $k=(n-d)/c$ of variables to write

$$\sum_{\substack{N<n\leqslant N_2\\ n\equiv d \bmod c}} e\left(\mu\cdot n^3\right)=\sum_{(N-d)/c<k\leqslant (N_2-d)/c} e\left(\mu\cdot (ck+d)^3\right).$$

Now, we apply the truncated Poisson summation formula, Lemma 16, to get

$$\sum_{(N-d)/c<k\leqslant (N_2-d)/c} e\left(\mu\cdot (ck+d)^3\right)$$

$$=\sum_{H_1\leqslant H\leqslant H_2}\int\limits_{(N-d)/c}^{(N_2-d)/c} e\left(\mu\cdot (cx+d)^3-hx\right)\ dx+O\left(\log(H_2-H_1)\right)$$

for suitable $H_1,H_2\in\mathbb{Z}$. By another linear change of variables $y=cx+d$, it follows that

$$\sum_{(N-d)/c<k\leqslant (N_2-d)/c} e\left(\mu\cdot (ck+d)^3\right)$$

$$=\frac{1}{c}\cdot e\left(\frac{hd}{c}\right)\cdot\sum_{H_1\leqslant H\leqslant H_2}\int\limits_{N}^{N_2} e\left(\mu\cdot y^3-\frac{h}{c}\cdot y\right)\ dy+O\left(\log(H_2-H_1)\right).$$

Combining everything and re-arranging summations, we obtain

$$\sum_{N<n\leqslant N_2} e\left(\frac{b+2al}{c}\cdot n+\frac{a}{c}\cdot n^2+\mu\cdot n^3\right)$$

$$=\sum_{H_1\leqslant h\leqslant H_2}\left(\sum_{d=1}^{c} e\left(\frac{b+2al+h}{c}\cdot d+\frac{a}{c}\cdot d^2\right)\right)\cdot\int\limits_{N}^{N_2} e\left(\mu\cdot y^3-\frac{h}{c}\cdot y\right)\ dy+\text{ error.}$$

7.5. **Remainder of the method.** So far, the method is not difficult to understand. Here comes the point where it gets complicated and the fun starts. The reader is invited to study the remainder of the method in detail. We say only a few sentences about it. This part of the method has since been refined further, in particular by Huxley [7] and Bourgain [3].

The complete quadratic Gauss sums

$$S(a,b+2al+h;c)=\sum_{d=1}^{c} e\left(\frac{b+2al+h}{c}\cdot d+\frac{a}{c}\cdot d^2\right)$$

can be evaluated explicitly (see [4], for example). In particular, Kloosterman fractions of the form $\overline{a}/c$, where $a\overline{a}\equiv 1 \bmod c$, come into play. For the cubic exponential integrals

$$\int\limits_{N}^{2N} e\left(\mu\cdot y^3-\frac{h}{c}\cdot y\right)\ dy,$$

we also have an explicit evaluation using stationary phase. It turns out that cubic semipowers $h^{3/2}$ come into play. Eventually, we are led to sums of the form

$$\sum_c \sum_a \sum_l \left| \sum_h e\left(\frac{\overline{a}}{c} \cdot \frac{h^2}{4} + \frac{\overline{a}b - 2l + c\eta}{c} \cdot \frac{h}{2} - \frac{\nu h^{3/2}}{c} \right) \right|,$$

where

$$\nu = \frac{4}{3} \cdot \left(2cf'''(m(a,c))\right)^{-1/2}$$

and η is a real number which does not depend on the variables of summation. Using Hölder's inequality, the above sum is bounded in terms of

$$\sum_c \sum_a \sum_l \left| \sum_h e\left(\frac{\overline{a}}{c} \cdot \frac{h^2}{4} + \frac{\overline{a}b - 2l + c\eta}{c} \cdot \frac{h}{2} - \frac{\nu h^{3/2}}{c} \right) \right|^4.$$

After expanding the fourth power, this reduces to bounding certain bilinear forms which is carried out using a tool called double large sieve. One is left with a complicated spacing problem which requires counting variables in certain ranges and satisfying certain inequalities and equations. In particular, one needs to count 8-tuples $(h_1, ..., h_8)$ of integers satisfying the system

$$\begin{aligned} \sum_{j=1}^{4} \left(h_j^2 - h_{j+4}^2\right) &= 0 \\ \sum_{j=1}^{4} (h_j - h_{j+4}) &= 0 \\ \sum_{j=1}^{4} \left(h_j^{3/2} - h_{j+4}^{3/2}\right) &\ll G \end{aligned}$$

for some $G > 0$. After solving the said counting problems and choosing the parameter N suitably, one obtains the desired bound

$$\sum_{M < m \leqslant M_1} e\left(f(n)\right) \ll L^{9/56+\varepsilon} M^{37/56+\varepsilon},$$

where L is the size of $f'(x)$ on $(M, M_1]$.

References

[1] D. Bump, *Automorphic Forms and Representations*, Cambridge University Press (1997).

[2] E. Bombieri, H. Iwaniec, *On the order of* $\zeta 1/2 + it$, Annali Scuola Normale Superiore - Pisa 13 (1986) 449-472.

[3] J. Bourgain, *Decoupling, exponential sums and the Riemann zeta function*, Journal of the American Mathematical Society, arXiv:1408.5794v2.

[4] S.W. Graham, G. Kolesnik, *Van der Corput's Method of Exponential Sums*, Cambridge University Press (2008).

[5] M.N. Huxley, N. Watt, *Exponential sums and the Riemann zeta function*, Proc. London Math. Soc. 57 (1988) 1-24.
[6] M.N. Huxley, *Area, Lattice Points and Exponential Sums*, Oxford Science Publications (1996).
[7] M.N. Huxley, *Exponential Sums and Lattice Points III*, Proc. London Math. Soc. 87 (2003) 5910-609.
[8] H.L. Montgomery, Ten Lectures on the Interface between Analytic Number Theory and Harmonic Analysis, American Mathematical Society (1994).

8. Tutorial problems

The main objects of the tutorials are to apply the material to the Gauss circle problem (counting lattice points enclosed by a circle) and to supply some proofs for lemmas not proved in the lectures.

Tutorial 1: 1.1 Let $r(n)$ be the number of ways of writing the natural number n as a sum of two squares. Prove that

$$r(n) = 4\sum_{d|n} \chi_4(d),$$

where χ_4 is the non-trivial character modulo 4.

1.2 (Gauss circle problem) For $x > 0$, let $C(x)$ be the number of standard lattice points included in a circle of radius $\sqrt{x}$, centered at the origin. Prove that

$$C(x) = \pi x + O\left(x^{1/2}\right).$$

Tutorial 2: 2.1 Prove that

$$\sum_{s \leqslant y} \frac{1}{s} = \log y + \gamma - \frac{\psi(y)}{y} + O\left(\frac{1}{y^2}\right)$$

as $y \to \infty$, where γ is the Euler constant and $\psi(y)$ is the sawtooth function, defined by $\psi(y) = y - [y] - 1/2$.

2.2 Let $C(x)$ be the function defined in problem 1.2. Prove that

$$C(x) = \pi x + R(x) + O(1),$$

where

$$R(x) = 4\sum_{d \leqslant \sqrt{x}} \left(\psi\left(\frac{x}{4d+1}\right) - \psi\left(\frac{x}{4d+3}\right) + \psi\left(\frac{x-3d}{4d}\right) - \psi\left(\frac{x-d}{4d}\right)\right) + O(1).$$

Tutorial 3: 3.1 Using Vaaler's trigonometric polynomial approximation for the function $\psi(x)$, express the error term $R(x)$, defined in problem 2.2, in terms of exponential sums.

3.2 Use the van der Corput bound to prove that
$$C(x) = \pi x + O\left(x^{1/3+\varepsilon}\right).$$

Tutorial 4: 4.1. Prove the Kusmin-Landau bound: Assume that f is a real-valued function which is continuously differentiable on the interval I which has a monotonic derivative and satisfies $||f'(x)|| \geqslant \delta > 0$ on I, where $||z||$ is the distance of z to the nearest integer. Then
$$\sum_{n \in I} e(f(n)) = O(\delta^{-1}).$$

4.2 Go through Vaaler's construction of a trigonometric polynomial approximation for the function $\psi(x)$.

Tutorial 5: 5.1. Prove the truncated Poisson summation formula: Suppose f is a real valued function which has two continuous derivatives on the interval $[a, b]$. Suppose also that f' is decreasing in $[a, b]$, that H_1 and H_2 are integers such that $H_1 < f'(x) < H_2$ for $a \leqslant x \leqslant b$ and that $H = H_2 - H_1 = 2$. Then
$$\sum_{a \leqslant n \leqslant b} e(f(n)) = \sum_{H_1 \leqslant h \leqslant H_2} \int_a^b e(f(n) - hx)\ dx + O(\log H).$$

5.2 Use the B-process to transform the exponential sums encountered in the Dirichlet divisor problem and the Gauss circle problem into new exponential sums.

Tutorial 6: 6.1 Combine the A- and B-processes to improve the exponent 1/3 in problem 3.2.

6.2 Use your knowledge to derive nontrivial bounds for sums of the form
$$\sum_{N < n \leqslant N'} n^{it},$$
where t is a real number and $0 < N < N' \leqslant 2N$. (Such sums come up in connection with bounds for the Riemann zeta function.)