Chapter 2

Research Talks by Young Researchers

1. Extremal Rays of the Betti Cones

Rajiv Kumar

Indian Institute of Technology Bombay

In 2008, Boij and Söderberg stated a conjecture for the extremal rays of the Betti cone of modules over a polynomial ring over a field. This was proved in 2009 by Eisenbud and Scheryer. A similar characterisation was proved in 2012 for a quadratic hypersurface of embedding dimension 2 by Berkesch, Burke, Erman and Gibbons. Generalising the second result, we find a class of one-dimensional rings where the extremal rays are exactly the ones spanned by the Betti tables of shifts of $R, R/(l)$ and $R/\mathfrak{m}^n$, where l is a regular element and $n \geq 1$. We also give an example of a one-dimensional ring to show that the above result does not hold in general.

2. Behaviour of Finiteness of the Set of Associated Primes of Lyubeznik Functor under Ring Extensions

Rajsekhar Bhattacharyya

Dinabandhu Andrews College, Garia, Kolkata

We study the behaviour of the finiteness of the set of associated primes of local cohomology modules, more generally of Lyubeznik functors, under various ring extensions. At first, we review the results for flat and faithfully flat extensions and we present new applications of them. Then, we show that the finiteness property can be transferred from a ring to its pure local subring. Further, we observed that under mild conditions on the rings, finiteness property can come down from cyclically pure ring extensions to its local base ring. In particular, we observe that the set of associated primes of Lyubeznik functors of a cyclically pure local subring (which turns out to be Cohen-Macaulay) of equicharacteristic or unramified regular local ring, is finite.

3. Asymptotic linear bounds of Castelnuovo-Mumford regularity in multi-graded modules

Dipankar Ghosh

Indian Institute of Technology Bombay

Let A be a Noetherian standard $\mathbb{N}$-graded algebra over an Artinian local ring A_0. Let $I_1, \ldots, I_t$ be homogeneous ideals of A and M a finitely generated $\mathbb{N}$-graded A-module. We prove that there exist two integers k_1 and k_1' such that

$$\operatorname{reg}(I_1^{n_1} \cdots I_t^{n_t} M) \leq (n_1 + \cdots + n_t)k_1 + k_1' \quad \text{for all} \quad n_1, \ldots, n_t \in \mathbb{N}.$$

We prove this result in a quite general set-up. As a consequence, we also obtain the following: If A_0 is a field, then there exist two integers k_2 and k_2' such that

$$\operatorname{reg}\left(\overline{I_1^{n_1}} \cdots \overline{I_t^{n_t}} M\right) \leq (n_1 + \cdots + n_t)k_2 + k_2' \quad \text{for all} \quad n_1, \ldots, n_t \in \mathbb{N},$$

where $\overline{I}$ denotes the integral closure of an ideal I of A.

4. Decomposing Gorenstein Rings as Connected Sums

Jai Laxmi

Indian Institute of Technology Bombay

In 2012, Ananthnarayan, Avramov and Moore give a new construction of Gorenstein rings from two Gorenstein local rings, called their connected sum. Given a Gorenstein Artin ring, one would like to know whether it decomposes as a connected sum and if so, what are its components. We will first give conditions for decomposability, and will further show that indecomposable components appearing in the decomposition are unique upto isomorphism.

5. Local cohomology of multi-Rees algebras, joint reduction numbers and product of complete ideals

Parangama Sarkar

Indian Institute of Technology Bombay

We find conditions on local cohomology modules of multi-Rees algebras of multi-graded admissible filtrations of ideals which enable us to predict joint reduction numbers. As a consequence, we are able to prove a generalization of a result of Reid-Roberts-Vitulli in the setting of analytically unramified local rings for completeness of power products of complete ideals.

6. Hilbert Kunz Multiplicity and Hilbert Kunz density function of Segre product and tensor product of rings

Mandira Mondol

Tata Institute of Fundamental Research

For a Noetherian local $(R, \mathbf{m})$ or for a Noetherian graded k-algebra R and an ideal $I \subset R$ such that $\ell(R/I)$ is finite and characteristic of $R = p > 0$, the Hilbert-Kunz function is defined as $HK(R, I)(q) = \ell(R/I^{[q]})$, where $q = p^n$ and $I^{[q]}$ denotes the n^{th} Frobenius power of I. In early $80's$ Monsky has proved that, if d denotes the dimension of R then

$$HK(R, I)(q) = e_{HK}(R, I)q^d + O(q^{d-1}),$$

where $e_{HK}(R, I)$, called *HKmultiplicity*, is a positive real number. Over the years, Hilbert-Kunz multiplicity for different rings has been studied by Monsky, Conca, Brenner, Trivedi, Watanabe,Yoshida and many others. To study e_{HK}, in a recent paper Trivedi has introduced a new invariant for a pair (M, I), where R is a standard graded Noetherian ring of dimension d over an algebraically closed field k of char $p > 0$, I is an homogeneous ideal of finite colength and M is a finitely generated graded R-module, called the *Hilbert Kunz density function* of (M, I)[denoted $HKd(M, I)$]. I will discuss HKd functions, and as a consequence, the e_{HK} of Segre product and Tensor product of finitely many rings.

7. On Hilbert coefficients of parameter ideals and Cohen-Macaulayness

Kumari Saloni

Indian Institute of Technology Guwahati

Let $(R, \mathfrak{m})$ be a Noetherian local ring, Q a parameter ideal and K an $\mathfrak{m}$-primary ideal containing Q. The Hilbert coefficients of Q with respect to K, namely $g_i^K(Q)$ which arise from the function $\ell(R/KQ^n)$, contain considerable information about the properties of the ring R. We discuss a necessary and sufficient condition for an unmixed local ring R to be Cohen-Macaulay in terms of $g_0(Q)$ and $g_1(Q)$. As a consequence, we obtain a result of Ghezzi et al. which settles the negativity conjecture of W. V. Vasconcelos in unmixed local rings.

8. de Rham Cohomology of Local Cohomology modules

Rakesh Reddy

Indian Institute of Science Education and Research, Trivandrum

Let K be a field of characteristic zero and $R = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables with coefficients in K. Let I be an ideal in R. Let $H_I^i(R)$ denotes the i^{th} local cohomology module. Let $A_n(K)$ be the n^{th} Weyl algebra. For an $A_n(K)$ module N let $H^i(\partial; N)$ denote the de Rham cohomology module of N with respect to $\partial = \partial_1, \cdots, \partial_n$.

In one of his remarkable papers Lyubeznik proved that for each i, $H_I^i(R)$ are holonomic $A_n(K)$ modules. In general de Rham cohomology modules of a holonomic $A_n(K)$ module are only finite dimensional K vector spaces. Recently Tony J. Puthenpurakal related $\dim_K H^*(\partial; H_I^*(R))$ with some properties of the ideal I.

In this talk we study Euler-characteristic of curves, Cohen-Macaulay surfaces and non-singular surfaces. We relate Euler-characteristic of the curves with its degree. We also give a lower bound for $\dim_K H^n(\partial; H_P^{n-1}(R))$ in terms of number of points at infinity of $V(P)$, where P is a prime ideal of

height $n-1$. For Cohen-Macaulay surfaces we relate the Euler-characteristic with the cohomological dimension of the ideal. Finally we derive the Euler-characteristic of non-singular surfaces in terms of Bass numbers of local cohomology modules of the ideal.

9. Coisotropic Property of Characteristic Varieties

Jyoti Singh

Indian Institute of Technology Bombay

An important geometric invariant in the theory of D-modules is the characteristic variety. Purpose of this talk is to describe the coisotropic property of characteristic variety of a module over Weyl algebras with some examples. Let $A_n(\mathbb{C})$ be the Weyl algebra and M a finitely generated left $A_n(\mathbb{C})$-module. Let $Ch(M)$ denote the variety corresponding to the zeros of the annihilator of $gr^F M$ in $\mathbb{C}^{2n}$. It is an invariant of M, called its characteristic variety. We show that characteristic variety of a finitely generated $A_n(\mathbb{C})$-module with Bernstein is always coisotropic. In support of this, we give some examples of modules over Weyl algebra and their characteristic varieties which are coisotropic.

10. Castelnuovo-Mumford Regularity

Clare D'Cruz

Chennai Mathematical Institute

Does the regularity increase if we pass to the radical or remove embedded primes? By examples, we show that this happens. As a by-product we are also able to answer some related questions. In particular, we provide examples of licci ideals related to monomial curves in $\mathbf{P}^3$ (resp. in $\mathbf{P}^4$) such that the regularity of their radical is essentially the square (resp. the cube) of that of the ideal.